Harmonic oscillator force between heavy quarks

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A renormalization-group procedure for effective particles is applied to quantum chromodynamics of one flavor of quarks with a large mass m in order to calculate light-front Hamiltonians for heavy quarkonia H_λ using perturbative expansion in the coupling constant α_λ. λ is the renormalization-group parameter with the interpretation of an inverse of the spatial size of the color charge distribution in the effective quarks and gluons. The eigenvalue equation for H_λ couples quark-antiquark states with sectors of a larger number of constituents. The coupling to states with more than one effective gluon, and interactions in the quark-antiquark-gluon sector, are removed at the price of introducing an ansatz for the gluon mass μ^2. The simplified equation is used to evaluate a new Hamiltonian of order α_λ that acts only in the effective quark-antiquark sector and in the nonrelativistic limit turns out to contain the Coulomb term with Breit-Fermi corrections and a spin-independent harmonic oscillator term with a frequency $\omega=[(4/3)(\alpha_\lambda/\pi)]^{1/2}\lambda(\lambda/m)^2(\pi/1152)^{1/4}$. The latter originates from the hole excavated in the overlapping quark self-interaction gluon clouds by the exchange of effective gluons between the quarks. The new term is largely independent of the details of μ^2 and in principle can fit into the ballpark of phenomenology. The first approximation can be improved by including more terms in H_λ and solving the eigenvalue equations numerically.

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I. INTRODUCTION

The purpose of this article is to describe a procedure that starts from quantum chromodynamics (QCD) with only one flavor of massive quarks and produces the Schrödinger equation for heavy quarkonia in a single formulation of the theory. Only the first approximation for the final Hamiltonian is evaluated. In this simplest version, the procedure involves a guess for the gluon mass term. But the guess appears to have little influence on the result. The procedure itself is not limited to the simplest version and the gluon mass ansatz can be tested in the future in refined calculations and phenomenology. The procedure is relativistic and can be used for quarkonia in arbitrary motion, which is a prerequisite for application in high-energy processes. Chiral symmetry is explicitly broken in the case of heavy quarks and the issue of the spontaneous breaking of the symmetry is ignored.

The approach described here stems from the similarity renormalization-group procedure for Hamiltonians [1], which has been applied to QCD [2] using the light-front (LF) form of dynamics [3]. A new ingredient is the boost-invariant creation and annihilation operator calculus for effective quarks and gluons (see below). Otherwise, the LF approach has been known for a long time, mainly as a candidate for connecting two qualitatively different models of hadrons: the parton model in the infinite momentum frame (IMF) [4,5] and the constituent model in the rest frame of a hadron [6]. Many contributions in that area [7] have followed the seminal work on exclusive processes [8]. Through the boost invariance and precisely defined notion of effective constituents, the approach described here aims at eventually providing a bridge between the two models of hadrons in a single theoretical framework. Before this can happen, however, the case of a heavy quarkonium is chosen here as the simplest one to begin with and test the method. This does not mean that the structure of theoretical heavy quarkonia closely resembles the structure of nucleons or that one can study the parton model or the connection between the current and constituent u and d quarks in the context of the heavy quarkonium discussed here. But our formulation that in principle can be attempted in these cases is greatly simplified by the restriction to only one heavy flavor. Even in this simplified case the constituent picture faces a number of problems when one attempts to formulate it *ab initio* in quantum field theory, where local interactions require cutoffs that dwarf any finite mass parameters and one is forced to think about infinitely many bare particles in the Fock space to begin with. Thus it is important to explain how the new procedure works in this case before one tackles the more complex theory with light fermions.

In the LF dynamics, the evolution of states is traced along the direction of a lightlike four-vector n^μ. With the conventional choice of $n=(1,0,0,-1)$, the variable $nx=x^0+x^3 \equiv x^+$ plays the role of time while $P^-=P^0-P^3$ is the Hamiltonian. In order to define it for bare particles in QCD, one has to choose a gauge. No serious alternative exists to $nA=A^+=0$. But the equation $D_\mu F^{\mu+}=j^+$ implies a constraint that is analogous to the Gauss law and forces the Fourier components of A^- to contain inverse powers of the kinematical momentum k^+. Since k^+ ranges from 0 to infinity, the inverse powers of k^+ produce singularities in the region around zero.

One can impose a lower bound on k^+, such as $k^+>\delta^+$, to regulate the theory [2,9–11]. The parameter δ^+ becomes a smallest unit of momentum that any particle, physical or virtual, can carry in such discretized theories. But the fixed unit breaks the boost invariance required for connection between the IMF and the rest frame of any hadron. Namely, when some physically relevant P^+ is made large, the smallest allowed $x=\delta^+/P^+$ becomes small. In the IMF $P^+\to\infty$ and the same small-x divergences re-appear despite the presence of δ^+. The key singularity is related to the dx/x distribution of gluons in the parton model and seems to require a dynamical mechanism to remove. One cannot just vary δ^+ together with P^+ because boosts cannot change the cutoff in a quantum theory constructed *ab initio* [12].

There exists a possibility that the small-k^+ singularities are related to the properties of the vacuum state. The sum rules for heavy quarkonia [13] include quark and gluon condensates [14] that may participate in the dynamics in the small-k^+ region [2,5,15–20]. In the QCD picture with such nontrivial ground state [21,22], and relativistic bound-state excitations of this state in the form of π mesons, one can hardly hope to resolve the small-k^+ singularity easily. The situation simplifies a lot in the case of quarks with mass $m \gg \Lambda_{QCD}$. The small-x singularity and a nonperturbative binding mechanism for quarks and gluons can interplay with each other without interference from the vacuum. It is not clear yet if similar dynamical effects in the small-x region, still well separated from the vacuum singularity itself, can play any significant role in the case of light quarks and gluons, and whether they can be relevant and contribute to the interactions and saturation mechanisms of partons [23–26].

The gluon mass ansatz is introduced to represent effects of the non-Abelian interactions. The ansatz is inserted at the level of solving the eigenvalue equation for the Hamiltonian H_λ, where λ is the renormalization-group (RG) parameter. It is a tool used to approximate the energy of interacting effective gluons in the presence of quarks and should not be associated with a violation of gauge symmetry in the initial Lagrangian. All the Hamiltonians discussed here are defined only in the gauge $A^+=0$, and H_λ does not exhibit local gauge invariance. The entire procedure that starts from the initial Lagrangian and eventually uses the mass ansatz for effective gluons in solving for eigenstates of H_λ is similar to the one proposed in Ref. [2] and later discussed in simplified matrix models [27,28]. New elements of the present approach are the limitation of power counting to the relative-motion variables, the exact boost invariance, removal of small-x divergences from the eigenvalue problem for H_λ through the mass ansatz as a function of the relative momenta, and no need for *ad hoc* potentials in the first approximation. All these features will be described in detail later.

In brief, $H_\lambda = T_\lambda + V_\lambda$, where T_λ is the kinetic energy operator made of all terms that are bilinear in the creation and annihilation operators for the effective particles, and V_λ represents all other terms. The parameter λ defines the width of momentum-space form factors in V_λ. For some value of $\lambda=\lambda_0\sim 1$ GeV, one can freely add to H_{λ_0} a term of the form $[1-(\alpha_{\lambda_0}/\alpha_s)^n]\mu^2$, where $n \geqslant 2$. The number 2 ensures that corrections to the first approximation occur first in the fourth-order of expansion of H_{λ_0} in powers of $g_0 = g_{\lambda_0}$, $\alpha_0 = g_0^2/(4\pi)$. This is the lowest order at which a perturbative shift in the gluon energy in any state can influence the contribution of that state to the dynamics of any other state in perturbation theory, keeping intact the QED-like small coupling expansion scheme with a Coulomb potential. μ^2 stands for the mass term that one assigns to the effective gluons of the transverse size $1/\lambda_0$. The interpretation of $1/\lambda$ as the spatial size of the color charge distribution in the corresponding effective particles is based on the feature mentioned above that V_λ contains vertex form factors of width λ in momentum space. The mass ansatz contributes to the invariant masses through μ^2/x, where x is the fraction of the longitudinal momentum carried by the gluon. α_s denotes the large, relativistic value of the coupling constant in QCD at the scale λ_0 with a true Λ_{QCD} in this scheme. The term with μ^2 vanishes for $\alpha_0=\alpha_s$. Nevertheless, only μ^2 counts when the ratio α_0/α_s is small. Thus, in the weak coupling expansion,

$$H_{\lambda_0} = T_{\lambda_0} + \mu^2 + [V_{\lambda_0} - (\alpha_0/\alpha_s)^n \mu^2] \tag{1}$$

and the term $[V_{\lambda_0} - (\alpha_0/\alpha_s)^n \mu^2]$ is treated as a source of small corrections in comparison to μ^2.

The requirement of cancellation of the small-x singularities in the effective dynamics imposes some perturbatively determined constraints on the otherwise nonperturbative ansatz for μ^2. These constraints restrict the behavior of μ^2 as a function of the gluon motion with respect to other constituents. In the future, the refined versions of the same procedure may provide constraints that come closer to the actual behavior of gluons. This behavior is hoped to be uncovered in computer simulations that one may build around the first approximation. Then, the extrapolation to $\alpha_0=\alpha_s$ can recover the original theory from a few terms in the weak-coupling expansion if μ^2 approximates the behavior of effective gluons well. Initially, the gluon mass term is viewed as a function of the relative momenta and Λ_{QCD}. The latter depends on α_0 as $\lambda_0 \exp(-c/\alpha_0)$ with a positive constant c. This means that Λ_{QCD} vanishes to all orders in the perturbative expansion, and μ^2 is considered to be on the order of 1. Before one knows more, μ can only be estimated on the basis of implications for the resulting Schrödinger equation. The size of μ can be compared with four scales: Λ_{QCD}, λ_0, m, and the Bohr momentum scale, $k_B=\alpha_0 m/2$, which is distinguished nonperturbatively by the Coulomb interaction. But if μ^2 is right, then $[V-(\alpha_0/\alpha_s)^n\mu^2]$ must be a source of only small corrections in the whole range of couplings between 0 and α_s. This is taken for granted in the present article. Since the first approximation turns out to be not sensitive to the details of μ, new information can be obtained only in the refined calculations.

The value of the weak-coupling expansion scheme for Hamiltonians is that it starts from a local theory and leads to H_λ that is capable of describing physically relevant nonperturbative dynamics even if H_λ is calculated only in low orders. This idea is known to work in the case of QED: the Coulomb potential accounts for highly nonperturbative dynamics of atoms, including the nature of chemical bond, while the Hamiltonian itself is only of the formal order of α. Condensed-matter physics illustrates this point in still wider domain.

But when one applies the weak-coupling expansion idea to QCD [2], one faces the fact that the strong-coupling constant rises to 10, 30, or even 100 times larger value than in the case of atoms or positronium in QED. This leads to a complex interplay between the perturbative and nonperturbative parts of the calculation, enhanced dependence of observables on the RG parameter λ, problems with obtaining the Poincaré symmetry in solutions, and amplification of artifacts due to the small-k^+ regularization. Most of the problems seem to come from perturbation theory in the RG part of the calculation. An exact RG procedure by definition provides H_λ whose structure depends on λ but the spectra and S-matrix elements do not. However, when one uses expansion in powers of α_0 and then extrapolates to $\alpha_0=\alpha_s$, a considerable dependence of the eigensolutions on λ can ensue because of missing many terms. This is visible in models that are asymptotically free and produce bound states [28,29]. One has to find the right value for α_s at given λ_0 from fits to bound-state observables, and perform consistency checks for whole sets of different observables [30]. In QCD, such checks involve the unknown functions of momenta in the finite parts of the ultraviolet counterterms, unknown terms depending on Λ_{QCD}, and artifacts of the regularization of small-k^+ divergences. So many unknowns suggest the possibility that the approach may never achieve the desired level of predictive power. But the simplicity of the harmonic potential found here in the first approximation illustrates that there is a high degree of order in the rich structure of H_λ. It is a consequence of preserving all seven kinematical symmetries of the LF scheme in the RG procedure. These symmetries limit the large number of terms that are allowed by the LF power counting using absolute momentum variables [2], to a much smaller number of terms that depend only on the relative momenta of the constituents. The LF symmetries must also be respected by the initial regularization prescription for the required counterterms to be simple.

The renormalization-group procedure for effective particles (RGEP), which is employed here, begins with regularization of the ultraviolet and small-k^+ divergences by insertion of some regularization factors r only in the interaction terms in the initial Hamiltonian. Roughly speaking, the regularization factors have the structure $r=r_\Delta r_\delta$, for every bare particle in every vertex. All relevant details of the regularization factors are given in Sec. II and Appendix A. The factors r_Δ limit the range of relative transverse momenta of the interacting bare particles by the parameter Δ. These factors are responsible for the transverse ultraviolet finiteness of the regulated theory. Then, the RGEP procedure produces the effective H_λ by solving operator differential equations. As a result, H_λ is written in terms of operators that only create and annihilate effective particles, and interactions of these contain new form factors f_λ that differ considerably from the regularization factors r_Δ of the initial bare theory. The form factors contain only a finite width parameter λ and are completely independent of the regularization scheme. Also, f_λ limits changes of entire invariant masses of groups of effective particles in interaction. These invariant masses depend on the transverse and longitudinal momenta simultaneously in a relativistically invariant way. It turns out that by lowering λ through solution of the RGEP differential equations with initial conditions that include counterterms, one excludes a possibility that $r_\Delta\neq 1$ when $\lambda\ll\Delta$, because the relative transverse momenta are always smaller than the invariant masses. Thus, on the way of reducing λ from infinity (regulated bare theory, $f_\infty=1$) to a finite value (effective theory with f_λ), RGEP introduces some correlations of special relativity between transverse and longitudinal momenta in the effective interaction terms in H_λ.

In the case of the small-k^+ singularities, the regulating factors r_δ limit only the ratios of momenta k^+. The ratios are limited from below by the dimensionless parameter δ. As already mentioned, all details of the factors r_δ are explained later. But there exists a qualitative difference between the ultraviolet regularization r_Δ and the small-x regularization r_δ that should be pointed out in advance for conceptual reasons. Let us stress first that every creation or annihilation operator, labeled by momentum k^+, that enters the initial interaction Hamiltonian is supplied with a corresponding factor $r_\delta(x)$, where $x=k^+/p^+$, and p^+ is the sum of all momenta that label all creation operators, or, equivalently, all annihilation operators in the same interaction term (see Appendix A). The RGEP procedure does not remove dependence on r_δ from H_λ because the small-x singularities are not purely ultraviolet in nature. However, the effective theory does lead to cancellation of the small-x singularities in colorless states, which is a correlation that is built in through the gauge symmetry of the Lagrangian that was used to construct the initial Hamiltonian, and which is preserved in the RGEP procedure. A cross check on the effective theory is provided by the fact that in QCD with $r_\delta(x)\sim x^\delta$, H_λ contains the coupling constant g_λ, which depends on the scale λ in the same asymptotically free way [32] that characterizes the running coupling constant dependence on the renormalization scale in the Feynman diagrams [33,34]. This shows that there exists a regularization procedure of small-x singularities that together with RGEP renders correlations in the effective H_λ that correspond to the known behavior of the fully relativistic theory in its perturbative domain.

Hamiltonians H_{λ_0} with small λ_0 are worth studying because their eigenstates can be expanded in the effective particle basis in the Fock space and the wave functions in this expansion are expected to correspond to the constituent picture of hadrons. This is envisioned in analogy to the models based on Yukawa theory [35]. The interactions are suppressed by the form factors and cannot copiously create new constituents, even if the coupling constant α_0 becomes large. Exotic hadron states may have their probability distributions shifted in the number of effective particles above the constituent quark model values of 2 or 3 [36]. The effective dynamics can be in agreement with requirements of special relativity even if it is limited to a small number of effective constituents, and the RGEP provides rules for constructing the representation of the Poincaré group [37]. But the key feature is that the transition between the bare and effective degrees of freedom is made in one and the same formalism. There is no need to match different formulations of the theory, such as in the case of lattice theory and the continuum perturbation theory in the Minkowski space [38–40], with none of the parts meant to cover the whole range of relevant scales on its own.

One more comment is required concerning the small-x divergences in the eigenvalue equation for H_λ. In the initial studies that used δ^+ to limit bare particles' momenta and employed coupling coherence to derive certain H_λ [41–45], one could keep only a quark-antiquark sector in the corresponding eigenvalue problem and the resulting equation was finite in the limit $\delta^+ \to 0$. Similarly, no small-k^+ divergences were encountered in the case of gluonium approximated by states of only two gluons [46]. In contrast, the present RGEP approach demands inclusion of states that contain an additional effective gluon which is needed to cancel small-x divergences. For example, if one keeps only a pair of the effective quark and antiquark, the leading nonrelativistic (NR) terms are free from the small-x singularities [31] but relativistic corrections are singular [47,48]. When the additional gluon is included, the condition of cancellation of the small-x divergences becomes a guide in understanding the gluon dynamics. The rules of including the gluons must be brought under quantitative control and the well-known case of heavy quarkonia provides a laboratory for testing the approach based on the gluon mass ansatz. The tests require a first approximation to begin with and a candidate is identified in the next sections.

This paper is organized as follows. Section II describes the initial Hamiltonian of LF QCD with one heavy flavor and the procedure for deriving an effective H_λ. Section III discusses the eigenvalue equation for a quarkonium, and introduces the ansatz for the gluon mass term. Small-x effects in the dynamics are described in Sec. IV. The resulting potential in the Schrödinger equation for a $Q\overline{Q}$ bound state is described in Sec. V. Section VI provides a brief summary and outlook. The Appendixes contain key details required for completeness.

II. HAMILTONIANS

The regularized canonical Hamiltonian of LF QCD with one heavy flavor of quarks, H, is given in Appendix A. It includes ultraviolet counterterms. This section describes the main features of H and the RGEP derivation of the effective Hamiltonian H_λ with a finite width λ. H_λ is independent of the ultraviolet regularization factors r_Δ in H when $\Delta \to \infty$. The small-x regularization factors r_δ, which are also present in H, and their role in H_λ will be discussed later.

The initial Hamiltonian has the structure

$$H = H_{\psi^2} + H_{A^2} + H_{\psi A \psi} + H_{(\psi\psi)^2} + X, \tag{2}$$

where the term H_{ψ^2} denotes the kinetic energy operator for quarks, H_{A^2} the kinetic energy operator for gluons, $H_{\psi A\psi}$ is the interaction term that couples gluons to quarks, $H_{(\psi\psi)^2}$ is the instantaneous interaction between quarks, and X denotes all other terms including the counterterms.

In terms of the creation operators for bare particles, $b^\dagger$ for quarks, $d^\dagger$ for antiquarks, $a^\dagger$ for gluons, and the corresponding annihilation operators, the kinetic energy terms are of the form

$$H_{\psi^2} = \sum_{\sigma c} \int [k] \frac{k^{\perp 2} + m^2}{k^+} [b^\dagger_{k\sigma c} b_{k\sigma c} + d^\dagger_{k\sigma c} d_{k\sigma c}], \tag{3}$$

and

$$H_{A^2} = \sum_{\sigma c} \int [k] \frac{k^{\perp 2}}{k^+} a^\dagger_{k\sigma c} a_{k\sigma c}, \tag{4}$$

where k denotes the three kinematical momentum components, k^+ and $k^\perp = (k^1, k^2)$. The symbol in a bracket, such as $[k]$, refers to the integration measure,

$$[k] = \frac{dk^+ d^2k^\perp}{16\pi^3 k^+}. \tag{5}$$

The subscript c stands for color and σ for spin. The mass m is assumed to be very large in comparison to Λ_{QCD}.

The quark-gluon coupling terms in $H_{\psi A\psi}$ that preserve the number of quarks and antiquarks have the form

$$Y = g \sum_{123} \int [123] \tilde{r}_{3,1} [j_{23} b^\dagger_2 a^\dagger_1 b_3 - \overline{j}_{23} d^\dagger_2 a^\dagger_1 d_3 + \text{H.c.}]. \tag{6}$$

The regularization factor $\tilde{r}_{3,1}$ is singled out to indicate its presence. The coefficients j_{23} and $\overline{j}_{23}$, are functions of the quark and gluon colors, spins, and momenta, with all details provided in Appendix A. These coefficients contain the three-momentum conservation δ-function factors, denoted by $\tilde{\delta}$, color factors t^1_{23}, and products of spinors, $j^\mu_{23} = \overline{u}_2 \gamma^\mu u_3$ and $\overline{j}^\mu_{32} = \overline{v}_3 \gamma^\mu v_2$. The latter are contracted with polarization vectors for gluons, so that

$$j_{23} = \tilde{\delta} t^1_{23} g_{\mu\nu} j^\mu_{23} \varepsilon^{\nu *}_1, \tag{7}$$

and

$$\overline{j}_{23} = \tilde{\delta} t^1_{32} g_{\mu\nu} \overline{j}^\mu_{32} \varepsilon^{\nu *}_1. \tag{8}$$

The instantaneous term $H_{(\psi\psi)^2}$ contains

$$Z = -g^2 \sum_{1234} \int [1234] \tilde{\delta} t^a_{12} t^a_{43} [j^+_{12} \overline{j}^+_{34} / (k^+_1 - k^+_2)^2] \times [\tilde{r}_{1,2} \tilde{r}_{4,3} + \tilde{r}_{2,1} \tilde{r}_{3,4}] b^\dagger_1 d^\dagger_3 d_4 b_2. \tag{9}$$

The current factors j and the gluon polarization vectors grow with the relative transverse momenta of the interacting particles $\kappa^\perp$. These can increase to infinity and the regularization factors r_Δ are introduced to limit the range to a finite one. In addition, there are small-x divergences due to the inverse powers of x, especially in the gluon polarization vectors that contain terms proportional to $\kappa^\perp / x$. The small-x singularities are regulated by factors r_δ.

The RGEP procedure generates ultraviolet counterterms contained in the operator X in Eq. (2) and renders the effective particle Hamiltonian H_λ which is independent of r_Δ. The procedure is defined order by order in the formal expansion in powers of the bare coupling constant g. This expansion is eventually re-written in terms of the effective coupling constant g_λ, which replaces g in H_λ and depends on the ratio λ/Λ_{QCD} [32]. The procedure is designed so that energy differences in denominators of the perturbative evaluation of H_λ are limited from below by λ. Therefore no infrared divergences are encountered in the evaluation of H_λ. Also, no perturbative intrusion into the binding mechanism is generated when λ is kept above the scale of typical relative momenta of the bound-state constituents. These features qualify the RGEP as a candidate for providing an answer to the well-known question of how it is possible that a simple two-body Schrödinger equation may represent a solution to a theory as complex as QCD [2,49].

A very brief recapitulation of the RGEP is provided here for completeness. The derivation of H_λ begins with a unitary change of the degrees of freedom from the bare quarks and gluons in Eq. (2) to the effective ones. Let q commonly denote the bare operators $b^\dagger$, $d^\dagger$, and $a^\dagger$, and their Hermitian conjugates. The operators q are transformed by a unitary operator $\mathcal{U}_\lambda$ into operators q_λ that create or annihilate effective particles with identical quantum numbers,

$$q_\lambda = \mathcal{U}_\lambda q \mathcal{U}_\lambda^\dagger. \tag{10}$$

The bare pointlike particles in H of Eq. (2) correspond to λ equal infinity. One rewrites the Hamiltonian H in terms of q_λ and obtains

$$H = H_\lambda(q_\lambda). \tag{11}$$

Using $\mathcal{U}_\lambda$, one has

$$\mathcal{H}_\lambda \equiv H_\lambda(q) = \mathcal{U}_\lambda^\dagger H \mathcal{U}_\lambda. \tag{12}$$

Thus $\mathcal{H}_\lambda$ has the same coefficient functions in front of products of qs as the effective H_λ has in front of the unitarily equivalent products of q_λ's. Differentiating $\mathcal{H}_\lambda$ with respect to λ, one obtains

$$\mathcal{H}'_\lambda = -[\mathcal{T}_\lambda, \mathcal{H}_\lambda], \tag{13}$$

where $\mathcal{T}_\lambda = \mathcal{U}_\lambda^\dagger \mathcal{U}'_\lambda$. $\mathcal{T}_\lambda$ is constructed using the notion of vertex form factors for effective particles. For example, if an operator without a form factor has the structure

$$\hat{O}_\lambda = \int [123] V_\lambda(1,2,3) q_{\lambda 1}^\dagger q_{\lambda 2}^\dagger q_{\lambda 3}, \tag{14}$$

then the operator with a form factor is written as $f_\lambda \hat{O}_\lambda$ and has the structure

$$f_\lambda \hat{O}_\lambda = \int [123] f_\lambda(\mathcal{M}_{12}, \mathcal{M}_3) V_\lambda(1,2,3) q_{\lambda 1}^\dagger q_{\lambda 2}^\dagger q_{\lambda 3}. \tag{15}$$

Different choices of the function f_λ imply different interactions. The choice adopted in this study is [32]

$$f_\lambda(\mathcal{M}_{12}, \mathcal{M}_3) = \exp[-(\mathcal{M}_{12}^2 - \mathcal{M}_3^2)^2/\lambda^4)]. \tag{16}$$

For any operator $\hat{O}$ expressible as a linear combination of products of creation and annihilation operators, $f\hat{O}$ contains a form factor $f_\lambda(\mathcal{M}_c, \mathcal{M}_a)$ in front of every product. $\mathcal{M}_c$ and $\mathcal{M}_a$ stand for the total free invariant masses of the particles created (subscript c) and annihilated (subscript a) by a given product.

The effective Hamiltonian is defined to have the structure

$$H_\lambda = f_\lambda G_\lambda, \tag{17}$$

where G_λ has to be calculated for given f_λ. One uses $\mathcal{G}_\lambda = G_\lambda(q)$, which is introduced in the same way as $\mathcal{H}_\lambda$ in Eq. (11). $\mathcal{G}_I$ satisfies the differential equation

$$\mathcal{G}'_I = [f\mathcal{G}_I, \{(1-f)\mathcal{G}_I\}'_{\mathcal{G}_0}], \tag{18}$$

where $\mathcal{G}_I = \mathcal{G} - \mathcal{G}_0$, $\mathcal{G}_0$ is the part of H that does not depend on the coupling constant g, and the curly bracket with subscript $\mathcal{G}_0$ denotes $\mathcal{T}$ that solves [31]

$$[\mathcal{T}, \mathcal{G}_0] = [(1-f)\mathcal{G}_I]'. \tag{19}$$

The initial condition for Eq. (18) is that $\mathcal{G}_\infty = H$,

$$\mathcal{G}_\lambda = H + \int_\infty^\lambda ds [f_s \mathcal{G}_{Is}, \{(1-f_s)\mathcal{G}_{Is}\}_{\mathcal{G}_0}]. \tag{20}$$

This equation shows that one can find the counterterms X in H that remove regularization dependence from $\mathcal{G}_\lambda$. $\mathcal{H}_\lambda = f_\lambda \mathcal{G}_\lambda$ and H_λ is obtained by replacing q by q_λ.

$\mathcal{G}_{I\lambda}$ is expanded into a series of terms $\tau_n \sim g^n$,

$$\mathcal{G}_I = \sum_{n=1}^{\infty} \tau_n. \tag{21}$$

τ_1 is independent of λ. Only the term $H_{\psi A \psi}$ needs to be discussed here. According to Eq. (6), $\tau_1 = \alpha_{21} + \alpha_{12}$, where α_{21} denotes terms that create a gluon and α_{12} the terms that annihilate a gluon (the left subscript denotes the number of creation and the right subscript the number of annihilation operators). The corresponding effective Hamiltonian interaction term is obtained by multiplying the integrand in Eq. (6) by f_λ and transforming q's into q_λ's.

When one neglects the terms that change the number of particles by more than 1, $\tau_2 = \beta_{11} + \beta_{22}$. Equation (18) implies

$$\tau'_2 = [\{f'\tau_1\}, f\tau_1] \equiv f_2[\tau_1 \tau_1], \tag{22}$$

with $f_2 = \{f'\}f - f\{f'\}$. The first factor f in f_2 refers to invariant masses in the first τ in the square bracket, and the second f in f_2 is for the second τ. The square bracket denotes all connected terms that result from contractions that replace products $q_i q_j^\dagger$ by commutators or anticommutators appropriately for bosons and fermions. The solution for τ_2 is then given by

$$\tau_{2\lambda} = \mathcal{F}_{2\lambda}[\tau_1 \tau_1] + \tau_{2\infty}, \tag{23}$$

where $\tau_{2\infty}$ includes $H_{(\psi\psi)^2}$ and the second-order mass counterterms from X in Eq. (2). $\mathcal{F}_{2\lambda}$ depends on incoming and outgoing momenta in the two vertices generated by the τ's. If one labels the three successive configurations of particle momenta by a, b, and c, in the sequence $a\tau_{ab}b\tau_{bc}c$, and introduces the symbol $uv=\mathcal{M}_{uv}^2-\mathcal{M}_{vu}^2$, where $\mathcal{M}_{uv}^2$ denotes the free invariant mass of a set of particles from the configuration u that are connected to the particles in the configuration v by the interaction τ_{uv} in the sequence $u\tau_{uv}v$, the vertex form factor of Eq. (16) in the interaction τ_{uv} can be written as

$$f_\lambda(\mathcal{M}_{ab},\mathcal{M}_{ba})=\exp[-(ab^2/\lambda^4)]\equiv f_{ab}. \tag{24}$$

If one then denotes the parent momentum for the vertex τ_{uv} by P_{uv}, and writes p_{uv} in place of P_{uv}^+, while all the minus components of momenta of the virtual quarks and gluons are given by the eigenvalues of $\mathcal{G}_0=H_{\psi^2}+H_{A^2}$,

$$\mathcal{F}_2(a,b,c)=\frac{p_{ba}ba+p_{bc}bc}{ba^2+bc^2}[f_{ab}f_{bc}-1]. \tag{25}$$

The second-order perturbation theory renders

$$H_\lambda=T_{q\lambda}+T_{g\lambda}+f_\lambda[Y_{qg\lambda}+V_{q\bar{q}\lambda}+Z_{q\bar{q}\lambda}]. \tag{26}$$

The kinetic energy term for effective quarks is

$$T_{q\lambda}=\sum_{\sigma c}\int[k]\frac{k^{\perp 2}+m_\lambda^2}{k^+}[b_{\lambda k\sigma c}^\dagger b_{\lambda k\sigma c}+d_{\lambda k\sigma c}^\dagger d_{\lambda k\sigma c}], \tag{27}$$

where

$$m_\lambda^2=m_0^2+(4/3)g^2\int[x\kappa]\tilde{r}_\delta^2(x)\sum_{12}|j_{23}^\nu\varepsilon_{\nu 1}^*|^2 \times[\mathcal{F}_{2\lambda}(m^2,\mathcal{M}^2,m^2)-\mathcal{F}_{2\lambda_0}(m^2,\mathcal{M}^2,m^2)]/k_3^+. \tag{28}$$

In the order of appearance, m_0^2 is the quark mass squared that should be present in $T_{q\lambda_0}$ in order to fit data for quarkonia, $m_0^2=m^2+o(g^2)$. The factor 4/3 comes from color, $(N_c^2-1)/(2N_c)$. The integration measure is

$$[x\kappa]=dx\,d^2\kappa^\perp/[16\pi^3x(1-x)], \tag{29}$$

where $x=k_1^+/k_3^+$ is the fraction of the quark momentum k_3 carried by the virtual gluon, and $\kappa^\perp=k_1^\perp-xk_3^\perp$ is the relative transverse momentum of the gluon with respect to the quark 2. The effective mass does not depend on the particle motion. This is a unique property of the RGEP in LF dynamics. The small-x regularization factor is

$$\tilde{r}_\delta(x)=r_\delta(x)r_\delta(1-x), \tag{30}$$

where [32]

$$r_\delta(x)=x^\delta\theta(x). \tag{31}$$

The middle argument of $\mathcal{F}_{2\lambda}$ is

$$\mathcal{M}^2=(m^2+\kappa^{\perp 2})/(1-x)+\kappa^{\perp 2}/x, \tag{32}$$

and

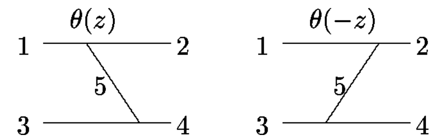

FIG. 1. Momentum labels in the interaction term mediated by the exchange of one high-energy gluon. The same labeling is used in the exchange of effective low-energy gluon in the next section and the appendixes.

$$\mathcal{F}_{2\lambda}(m^2,\mathcal{M}^2,m^2)/k_3^+=[f_\lambda^2(\mathcal{M}^2,m^2)-1]/(\mathcal{M}^2-m^2). \tag{33}$$

The gluon kinetic energy term reads

$$T_{g\lambda}=\sum_{\sigma c}\int[k]\frac{k^{\perp 2}+\mu_\lambda^2}{k^+}a_{\lambda k\sigma c}^\dagger a_{\lambda k\sigma c}. \tag{34}$$

The explicit form of μ_λ^2 [31,32] is not needed here.

The next term in Eq. (26) is $Y_\lambda=f_\lambda Y_{qg\lambda}$,

$$Y_\lambda=g\sum_{123}\int[123]r_\delta(x_{1/3})r_\delta(x_{2/3})f_\lambda(\mathcal{M}_{12}^2,m^2) \times[j_{23}b_{\lambda 2}^\dagger a_{\lambda 1}^\dagger b_{\lambda 3}+\overline{j_{23}}d_{\lambda 2}^\dagger a_{\lambda 1}^\dagger d_{\lambda 3}+\text{H.c.}]. \tag{35}$$

The effective potential term, $V_\lambda=f_\lambda V_{q\bar{q}\lambda}$, originates from the exchange of bare gluons with jumps in the invariant mass of intermediate states above λ; see Fig. 1:

$$V_\lambda=-g^2\sum_{1234}\int[1234]\tilde{\delta}t_{12}^at_{43}^aV_\lambda(13,24)b_1^\dagger d_3^\dagger d_4b_2, \tag{36}$$

where

$$V_\lambda(13,24)=\frac{d_{\mu\nu}(k_5)}{k_5^+}j_{12}^\mu\overline{j_{43}^\nu}f_\lambda(\mathcal{M}_{13}^2,\mathcal{M}_{24}^2) \times[\theta(z)\tilde{r}_\delta(x_{5/1})\tilde{r}_\delta(x_{5/4})\mathcal{F}_{2\lambda}(1,253,4) +\theta(-z)\tilde{r}_\delta(x_{5/3})\tilde{r}_\delta(x_{5/2})\mathcal{F}_{2\lambda}(3,154,2)]. \tag{37}$$

The sum over polarizations of the intermediate gluon reads

$$d_{\mu\nu}(k_5)=-g_{\mu\nu}+\frac{n^\mu k_5^\nu+k_5^\mu n^\nu}{k_5^+}, \tag{38}$$

where the gluon momentum is

$$k_5^{+,\perp}=\varepsilon(z)(k_1^{+,\perp}-k_2^{+,\perp}), \tag{39}$$

and $\varepsilon(z)=\theta(z)-\theta(-z)$,

$$z=(k_1^+-k_2^+)/(k_1^++k_3^+), \tag{40}$$

while $x_5=|z|=k_5^+/(k_1^++k_3^+)$, and

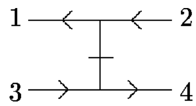

FIG. 2. Momentum labels in the instantaneous gluon interaction term.

$$k_5^- = k_5^{\perp 2}/k_5^+ . \tag{41}$$

The last term in Eq. (26) is the instantaneous interaction between effective quarks, $Z_\lambda = f_\lambda Z_{q\bar{q}\lambda}$,

$$Z_\lambda = -g^2 \sum_{1234} \int [1234]\tilde{\delta} t_{12}^a t_{43}^a Z_\lambda(13,24) b_{\lambda 1}^\dagger d_{\lambda 3}^\dagger d_{\lambda 4} b_{\lambda 2}, \tag{42}$$

where (see Fig. 2)

$$Z_\lambda(13,24) = \frac{1}{k_5^{+2}} j_{12}^+ \bar{j}_{34}^+ f_\lambda(\mathcal{M}_{13}^2, \mathcal{M}_{24}^2)[\theta(z)\tilde{r}_\delta(x_{5/1})\tilde{r}_\delta(x_{5/4}) + \theta(-z)\tilde{r}_\delta(x_{5/3})\tilde{r}_\delta(x_{5/2})]. \tag{43}$$

III. EIGENVALUE EQUATION

Once λ is lowered in perturbation theory to some value λ_0 just above the scale of binding mechanism, the resulting H_{λ_0} can produce the mass and wave function of a bound state of interest in a numerical diagonalization. The basis states can be limited to only those that have free invariant masses within a range of size about λ around the eigenvalue. This has been verified numerically in a matrix model with asymptotic freedom and bound states [28–30]. In that calculation, H_λ was derived using perturbation theory up to sixth order. A quite small set of effective basis states with energies between 4 MeV and 4 GeV was sufficient to reach accuracy close to 1% in the computation of the bound-state energy on the order of 1 GeV. In great contrast, the initial Hamiltonian of the model coupled all states in the entire range between 0.5 keV and 65 TeV. Preliminary estimates performed in Yukawa-like theories also indicate that the form factors f_λ suppress large momentum changes so strongly that the effective dynamics derived in low-order perturbation theory receives only small corrections from higher-order terms, even when the coupling constant is made comparable with 1 [35]. In the case of heavy quarkonia the same strategy should work even more accurately than in the Yukawa theory because α_s may be very small in comparison to 1. But the weak-coupling expansion for H_{λ_0} produces new interaction terms already in order α_0. These are derived here. An ansatz for the gluon mass allows us then to finesse the structure of the first approximation for the resulting $Q\bar{Q}$ potential.

The eigenvalue problem for H_λ reads

$$H_\lambda |P\rangle = E|P\rangle, \tag{44}$$

where $|P\rangle$ denotes an eigenstate of the operators P_λ^+ and $P_\lambda^\perp$ with their eigenvalues denoted by P^+ and $P^\perp$ (an example of the RGEP construction of the Poincaré algebra at scale λ in quantum field theory is given in Ref. [37]). The eigenvalue E has the form

$$E = (M^2 + P^{\perp 2})/P^+ . \tag{45}$$

The center-of-mass motion is separated from the binding mechanism, which is a unique LF-dynamics feature preserved by the RGEP, and P^+ and $P^\perp$ drop out of the eigenvalue equation. The state $|P\rangle$ is written in the effective particle basis as

$$|P\rangle = |Q_\lambda \bar{Q}_\lambda\rangle + |Q_\lambda \bar{Q}_\lambda g_\lambda\rangle + \cdots . \tag{46}$$

The dots denote components with more than three effective particles. Such expansion does not apply in the case of the bare particles because those interact locally and the interactions disperse probability density to high momentum regions and multiparticle sectors [35]. The wave function $\psi_{13}(\kappa_{13}^\perp, x_1)$ of the effective valence component $|Q_\lambda \bar{Q}_\lambda\rangle$, is introduced by the formula

$$|Q_\lambda \bar{Q}_\lambda\rangle = \int [13] P^+ \tilde{\delta} \psi_{13}(\kappa_{13}^\perp, x_1) b_{\lambda 1}^\dagger d_{\lambda 3}^\dagger |0\rangle, \tag{47}$$

where the quark and antiquark quantum numbers are labeled with 1 and 3, respectively. $\psi_{13}(\kappa_{13}^\perp, x_1)$ must have dimension of $1/\kappa_{13}^\perp$ for the canonical normalization condition to give $\langle P'|P\rangle = P^+ \tilde{\delta}(P-P')$ (quantum numbers of the states $|P\rangle$ and $|P'\rangle$ must be the same). The relative transverse momentum of two particles, 1 and 3, is always defined as

$$\kappa_{13}^\perp = (k_3^+ k_1^\perp - k_1^+ k_3^\perp)/(k_1^+ + k_3^+), \tag{48}$$

and $x_1 = k_1^+/P^+ = 1 - x_3$. The wave function depends on λ and quickly vanishes for $|\kappa_{13}^\perp| > \lambda$. The normalization condition gives

$$\langle Q_\lambda \bar{Q}_\lambda | Q_\lambda \bar{Q}_\lambda\rangle = N_{Q\bar{Q}}(\lambda) P^+ \tilde{\delta}(P-P')|_{P'=P}, \tag{49}$$

where

$$N_{Q\bar{Q}}(\lambda) = \sum_{13} \int [x_1 \kappa_{13}] |\psi_{13}(\kappa_{13}^\perp, x_1)|^2. \tag{50}$$

The probability of finding other components than the $|Q_\lambda \bar{Q}_\lambda\rangle$ is given by $1 - N_{Q\bar{Q}}(\lambda)$. The value of $N_{Q\bar{Q}}(\infty)$ is not known but it may be close to 0. On the other hand, one expects $N_{Q\bar{Q}}(\lambda)$ to be close to 1 when $m \gg \Lambda_{QCD}$ and

$$m \gg \lambda \gg \Lambda_{QCD} . \tag{51}$$

When the wave function is negligible for relative momenta much larger than such λ, the NR approximation must be accurate in description of the relative motion of quarks. In addition, when $\alpha_\lambda \ll 1$, the Coulomb binding mechanism is expected to work and the dominant region of momenta should lie around the Bohr momentum scale $k_B = \alpha_\lambda m$, provided that $\lambda \gg k_B$. At the same time, all the fermion spin and relativistic correction factors cannot become large (or diverging [51]) in the NR expansion because of the presence of f_λ [35]. But the dynamics of the dominant $|Q_\lambda \overline{Q}_\lambda\rangle$ component receives some significant contributions from the $|Q_\lambda \overline{Q}_\lambda g_\lambda\rangle$ component in the small-x_5 region, since the coupling to the gluon sector grows like $\kappa_5^\perp/x_5$ when $x_5 \to 0$. The gluon component may have a negligible contribution to the norm but has to be accounted for when x_5 is small.

If one neglected sectors with gluons entirely, the eigenvalue Eq. (44) would read

$$[T_{q\lambda} + f_\lambda (V_{q\bar{q}\lambda} + Z_{q\bar{q}\lambda})]|Q_\lambda \overline{Q}_\lambda\rangle = E|Q_\lambda \overline{Q}_\lambda\rangle. \quad (52)$$

This equation is mentioned here because an analogous one was considered before [43–45] in a scheme using coupling coherence and the absolute lower bound on gluon momenta, $k_5^+ > \delta^+$. The equation found in Ref. [43] had a finite limit when $\delta^+ \to 0$. The resulting dynamics contained a logarithmically rising potential and reproduced some of the characteristic features of the charmonium and bottomonium spectra. This was a considerable success in view of how crude the approximations were and the fact that the potential derived in Ref. [43] and employed in Refs. [44,45] behaved differently in the transverse and longitudinal directions. But that strategy could not work in the RGEP approach.

Three reasons can be given now for why the pure $|Q_\lambda \overline{Q}_\lambda\rangle$ approximation is not allowed in solving the eigenvalue problem for H_λ. Two of them are related to the fact that the coupling between the sectors $|Q_\lambda \overline{Q}_\lambda\rangle$ and $|Q_\lambda \overline{Q}_\lambda g_\lambda\rangle$ is proportional to the first power of the coupling constant g, being mediated by the term Y_λ of Eq. (35). The first argument is nonperturbative. It is based on the result that the matrix models with asymptotic freedom and bound states lead to a successful approximation to the eigenvalue equation with H_λ of second order in g_λ if and only if all important matrix elements in the properly chosen energy window are accounted for. These certainly include matrix elements on the order of g [28,29]. The second argument is perturbative, and concerns the evaluation of the effective Hamiltonian that acts in the sector $|Q_\lambda \overline{Q}_\lambda\rangle$ alone. When the states $|Q_\lambda \overline{Q}_\lambda g_\lambda\rangle$ are lifted in energy by an amount of order 1, quantum transitions in the quark-antiquark sector that proceed through the states $|Q_\lambda \overline{Q}_\lambda g_\lambda\rangle$ are formally of order g^2 and must be included when one computes the quark-antiquark dynamics in a series of powers of g up to terms of the explicit order of α_λ. The third argument is based on the fact that Eq. (52) has a finite limit when $\delta \to 0$ only in the leading NR approximation [31]. Relativistic corrections contain singularities [47,48] and the additional gluon sector has to be taken into account to remove them.

The eigenvalue Eq. (44) implies that the $|Q_\lambda \overline{Q}_\lambda g_\lambda\rangle$ component satisfies the equation

$$[T_{q\lambda} + T_{g\lambda} + V_{q\bar{q}g\lambda} - E]|Q_\lambda \overline{Q}_\lambda g_\lambda\rangle = -Y_\lambda |Q_\lambda \overline{Q}_\lambda\rangle. \quad (53)$$

$V_{q\bar{q}g\lambda}$ denotes interactions with sectors with more than one gluon and/or additional quark-antiquark pairs, and the non-Abelian gluon-quark and quark-antiquark potentials of order α_λ. The interactions cause a shift in the gluon energy and make the eigenvalue equation differ from a similar one for positronium. The idea of seeking an ansatz for the shift and building a corresponding first approximation in the quark-antiquark sector may seem completely new, but it patterns QED with the exception that there one knows from the outset that the leading approximation to a hydrogen atom or positronium is given by a two-body Schrödinger equation with a Coulomb potential [50–52]. The NR lattice approach to heavy quarkonia [53,54] also starts from a two-body picture. The key argument is not theoretical but comes from the phenomenology of hadrons. Theoretically, an ansatz for the energy shift in the sector $|Q_\lambda \overline{Q}_\lambda g_\lambda\rangle$ is an attempt to harness the giant eigenvalue problem for H by turning it into the eigenvalue problem for H_λ and identifying the corrected Coulomb picture that may apply as a first approximation in QCD.

A practical way to increase the invariant mass of the three-body sector and preserve the kinematical symmetries of LF dynamics is to add a mass μ^2 to $T_{g\lambda}$, using the rules outlined in Sec. I; see Eq. (1). Since the divergence in the small-x region disappears in the case of positronium when one adds a sector with a massless photon, a gluon mass that approaches zero when $x_5 \to 0$ can remove the small-x_5 divergence in the quarkonium case. The rotational symmetry condition on μ^2 can be imposed by demanding that the resulting potential in the quark-antiquark sector is a rotationally symmetric function in the center-of-mass variables (see below).

Given a gluon mass ansatz, the whole eigenvalue problem for H_λ is limited to only two coupled equations,

$$(T_q + \tilde{T}_g)|Q\overline{Q}\rangle g + Y|Q\overline{Q}\rangle = E|Q\overline{Q}g\rangle,$$

$$Y|Q\overline{Q}g\rangle + [T_q + f(V_{q\bar{q}} + Z_{q\bar{q}})]|Q\overline{Q}\rangle = E|Q\overline{Q}\rangle \quad (54)$$

(the subscript λ is omitted). $\tilde{T}_{g\lambda}$ differs from $T_{g\lambda}$ of Eq. (34) by replacement of the perturbative μ_λ^2 by the ansatz μ^2. It is understood that μ^2 may depend on λ. All terms of order g^2 in the three-body sector are ignored because they do not contribute to the dynamics of the $|Q\overline{Q}\rangle$ component in second-order perturbation theory (see below). This dynamics is described by the Hamiltonian $H_{Q\bar{Q}}$ that acts only in the quark-antiquark sector. One should keep in $H_{Q\bar{Q}}$ terms of formal orders 1, g, and g^2, when the effective Hamiltonian H_λ is calculated up to the terms of order g^2, while $\mu^2 \sim 1$. The bare g is understood to go over in higher-order calculations to a suitably defined g_λ [30,32]. The perturbative expansion is applied only in the evaluation of $H_{Q\bar{Q}}$. Solution for the bound-state spectrum of $H_{Q\bar{Q}}$ is not perturbative.

To evaluate $H_{Q\bar{Q}}$ as a power series in g, one can introduce an operator R that expresses the three-body component through the two-body one,

$$|Q\overline{Q}g\rangle = R|Q\overline{Q}\rangle. \quad (55)$$

Since Y is of order g, R is expected to be at least of order g. If $\hat{\mathcal{P}}$ denotes the projector on the $|Q\overline{Q}\rangle$ sector, one has $R = (1-\hat{\mathcal{P}})R = R\hat{\mathcal{P}}$ and $\hat{\mathcal{P}}R = R(1-\hat{\mathcal{P}}) = 0$. The effective

Hamiltonian in the $|Q\bar{Q}\rangle$ sector is then given by the formula [55,56] (see also Ref. [35] concerning the context of RGEP)

$$H_{Q\bar{Q}}=\frac{1}{\sqrt{\hat{\mathcal{P}}+R^\dagger R}}(\hat{\mathcal{P}}+R^\dagger)H_\lambda(\hat{\mathcal{P}}+R)\frac{1}{\sqrt{\hat{\mathcal{P}}+R^\dagger R}}. \quad (56)$$

In the first order in g,

$$RT_q-(T_q+\tilde{T}_g)R=Y. \quad (57)$$

Consequently, the second-order expression for the matrix elements of $H_{Q\bar{Q}}$ between different states i and j in the $|Q\bar{Q}\rangle$ sector is

$$\langle i|H_{Q\bar{Q}}|j\rangle=\langle i|[T_q+f(V_{q\bar{q}}+Z_{q\bar{q}})]|j\rangle + \frac{1}{2}\langle i|Y\left(\frac{1}{E_j-T_q-\tilde{T}_g}+\frac{1}{E_i-T_q-\tilde{T}_g}\right)Y|j\rangle. \quad (58)$$

The effective eigenvalue equation for heavy quarkonia, $H_{Q\bar{Q}}|Q\bar{Q}\rangle=E|Q\bar{Q}\rangle$, takes the form

$$\left[\frac{\kappa_{13}^{\perp 2}+m_\lambda^2}{x_1x_3}+\frac{m_Y^2(1)}{x_1}+\frac{m_Y^2(3)}{x_3}-M^2\right]\psi(\kappa_{13}^\perp,x_1) -\frac{4}{3}\frac{g^2}{16\pi^3}\int\frac{dx_2\,d^2\kappa_{24}^\perp}{x_2x_4}v_\lambda(13,24)\psi(\kappa_{24}^\perp,x_2)=0, \quad (59)$$

where

$$v_\lambda(13,24)=V_\lambda(13,24)+Z_\lambda(13,24) +\frac{1}{2x_5}d_{\mu\nu}(k_5)j_{12}^\mu\overline{j_{43}^\nu}w_\lambda(13,24), \quad (60)$$

and

$$w_\lambda(13,24)=\Bigg\{\Bigg[\frac{\theta(z)\tilde{r}_\delta(x_{5/1})\tilde{r}_\delta(x_{5/4})f_\lambda(m^2,\mathcal{M}_{52}^2)f_\lambda(\mathcal{M}_{53}^2,m^2)}{(\kappa_{13}^{\perp 2}+m^2)/x_1-[(\kappa_{13}^\perp-\kappa_{24}^\perp)^2+\mu^2(2,5,3)]/x_5-(\kappa_{24}^{\perp 2}+m^2)/x_2} +\frac{\theta(-z)\tilde{r}_\delta(x_{5/3})\tilde{r}_\delta(x_{5/2})f_\lambda(m^2,\mathcal{M}_{54}^2)f_\lambda(\mathcal{M}_{51}^2,m^2)}{(\kappa_{13}^{\perp 2}+m^2)/x_3-[(\kappa_{24}^\perp-\kappa_{13}^\perp)^2+\mu^2(1,5,4)]/x_5-(\kappa_{24}^{\perp 2}+m^2)/x_4}\Bigg] +\Bigg[\frac{\theta(z)\tilde{r}_\delta(x_{5/1})\tilde{r}_\delta(x_{5/4})f_\lambda(m^2,\mathcal{M}_{52}^2)f_\lambda(\mathcal{M}_{53}^2,m^2)}{(\kappa_{24}^{\perp 2}+m^2)/x_4-[(\kappa_{13}^\perp-\kappa_{24}^\perp)^2+\mu^2(2,5,3)]/x_5-(\kappa_{13}^{\perp 2}+m^2)/x_3} +\frac{\theta(-z)\tilde{r}_\delta(x_{5/3})\tilde{r}_\delta(x_{5/2})f_\lambda(m^2,\mathcal{M}_{54}^2)f_\lambda(\mathcal{M}_{51}^2,m^2)}{(\kappa_{24}^{\perp 2}+m^2)/x_2-[(\kappa_{24}^\perp-\kappa_{13}^\perp)^2+\mu^2(1,5,4)]/x_5-(\kappa_{13}^{\perp 2}+m^2)/x_1}\Bigg]\Bigg\}. \quad (61)$$

The terms with $\theta(z)$ describe the emission of the gluon by the quark and absorption by the antiquark, while the terms with $\theta(-z)$ describe the gluon emission by the antiquark and absorption by the quark; see Fig. 1. The first square bracket corresponds to the first term in the large round bracket in Eq. (58), and the second bracket corresponds to the second term. The mass terms m_Y^2 originate from the emission and re-absorption of the effective gluon by the same quark, in which case both terms in the bracket of Eq. (58) are equal. The mass terms read

$$m_Y^2(1)=(4/3)g^2\int[x\kappa]\tilde{r}_\delta^2(x)f_\lambda^2(m^2,\mathcal{M}^2) \times\frac{j^\nu j^{\mu *}d_{\mu\nu}(k)/x_1}{(\kappa_{13}^{\perp 2}+m^2)/x_1-(\kappa_{13}^{\perp 2}+\mathcal{M}_1^2)/x_1}, \quad (62)$$

where

$$\mathcal{M}_1^2=[\kappa^{\perp 2}+\mu^2(1',5',3)]/x+(\kappa^{\perp 2}+m^2)/(1-x), \quad (63)$$

and

$$m_Y^2(3)=(4/3)g^2\int[x\kappa]\tilde{r}_\delta^2(x)f_\lambda^2(m^2,\mathcal{M}^2) \times\frac{\overline{j^\nu}\,\overline{j^{\mu}}{}^* d_{\mu\nu}(k)/x_3}{(\kappa_{13}^{\perp 2}+m^2)/x_3-(\kappa_{13}^{\perp 2}+\mathcal{M}_3^2)/x_3}, \quad (64)$$

where

$$\mathcal{M}_3^2=[\kappa^{\perp 2}+\mu^2(1,5',3')]/x+(\kappa^{\perp 2}+m^2)/(1-x). \quad (65)$$

$\mathcal{M}$ is given by Eq. (32). The subscript $1'$ denotes the intermediate quark and $5'$ denotes the intermediate gluon in the self-interaction of the effective quark 1, and, similarly, $3'$

and 5′ denote the intermediate antiquark and gluon in the self-interaction of the effective antiquark 3.

The gluon four-momentum k_5 in the sum over polarizations, i.e., in $d_{\mu\nu}(k_5)$ in Eq. (38), can be written as

$$k_5^\alpha = \varepsilon(z) q_{ij}^\alpha + n^\alpha [k_5^- - \varepsilon(z) q_{ij}^-]/2, \tag{66}$$

where ij refers to quarks 1 and 2, or antiquarks 3 and 4,

$$q_{ij} = k_i - k_j . \tag{67}$$

The quark momentum four-vectors are on the mass shell. Since the gluon connects two vertices, one momentum k_5 in $d_{\mu\nu}(k_5)$ is contracted with the current carried by the quark, and the other with the current of the antiquark. In the self-interactions, both momenta are contracted with the same current. The momentum k_5 contracted with current j_{ij}^α can be expressed through q_{ij}^α. But the current conservation implies that the terms proportional to q_{ij} give zero. Therefore one can replace Eq. (38) in the gluon exchange terms by

$$d_{\mu\nu}(k_5) = -g_{\mu\nu} + n_\mu n_\nu [k_5^- + \varepsilon(z) \times (k_2^- - k_1^- + k_3^- - k_4^-)/2]/k_5^+ . \tag{68}$$

In the quark self-interaction one has

$$d_{\mu\nu}(k_{5'}) = -g_{\mu\nu} + n_\mu n_\nu \frac{k_{5'}^- + k_{1'}^- - k_1^-}{k_{5'}^+}, \tag{69}$$

with an analogous result for the antiquark.

The terms with the metric $g_{\mu\nu}$ are regular in the small-x_5 region, while the terms with $n_\mu n_\nu$ are singular. The metric terms lead in the well-known way to the Breit-Fermi spin-dependent terms with a Coulomb potential. A discussion of the Breit-Fermi terms and gluons in the context of QCD can be found in Ref. [13] and references therein. The singular terms with $n_\mu n_\nu$ are independent of the quark spin. It is shown below that the latter generate the harmonic force between quarks when combined with the fermions' self-interactions, which are also independent of the spin. Thus the harmonic force appears without Breit-Fermi terms. This result sets the RGEP approach apart from the models employed in Refs. [13,57–63], where one had to guess whether a confining potential appeared with or without Breit-Fermi terms. The spin-independent harmonic force is akin in this respect to the lattice picture and the original charmonium model based on the Coulomb force [64–67].

In the explicit discussion of singular small-x features of Eq. (59) in the next section, all the $g_{\mu\nu}$ terms are omitted. The reader should keep their presence in mind until they are re-inserted in Sec. V. The symbols of mass, wave function, and potential are provided with a tilde as a reminder about the need to include the $g_{\mu\nu}$ terms. Also, expressions for the quark masses are simplified by considering from now on only $\lambda = \lambda_0$. The subscript 0 indicates that $\lambda = \lambda_0$. The ansatz for μ^2 is understood to correspond to λ_0.

With the $g_{\mu\nu}$ terms hidden and $\lambda = \lambda_0$, the eigenvalue equation reads

$$\left(\frac{\kappa_{13}^{\perp 2} + m_0^2}{x_1 x_3} + \frac{\tilde m_1^2}{x_1} + \frac{\tilde m_3^2}{x_3} - M^2 \right) \tilde\psi(\kappa_{13}^\perp, x_1) - \frac{4}{3} \frac{g^2}{16\pi^3} \times \int \frac{dx_2\, d^2\kappa_{24}^\perp}{x_2 x_4} \frac{j_{12}^+ j_{43}^+}{P^{+2}} \frac{1}{x_5^2} \tilde v_0(13,24) \tilde\psi(\kappa_{24}^\perp, x_2) = 0, \tag{70}$$

where

$$\tilde v_0(13,24) = f_{13,24}[k_5^- + \varepsilon(z)(k_2^- - k_1^- + k_3^- - k_4^-)/2] \times [\theta(z)\tilde r_{5/1}\tilde r_{5/4}\mathcal{F}_{1,253,4} + \theta(-z)\tilde r_{5/3}\tilde r_{5/2}\mathcal{F}_{3,154,2}] + f_{13,24}[\theta(z)\tilde r_{5/1}\tilde r_{5/4} + \theta(-z)\tilde r_{5/3}\tilde r_{5,2}] + \frac{1}{2}[k_5^- + \varepsilon(z)(k_2^- - k_1^- + k_3^- - k_4^-)/2] w_0, \tag{71}$$

and the last factor of $w_0 \equiv P^+ w_{\lambda_0}(13,24)$ is abbreviated to

$$w_0 = \frac{\theta(z)\tilde r_{5/1}\tilde r_{5/4} f_{1,52} f_{53,4}}{k_1^- - \tilde k_5^-(2,5,3) - k_2^-} + \frac{\theta(-z)\tilde r_{5/3}\tilde r_{5/2} f_{3,54} f_{51,2}}{k_3^- - \tilde k_5^-(1,5,4) - k_4^-} + \frac{\theta(z)\tilde r_{5/1}\tilde r_{5/4} f_{1,52} f_{53,4}}{k_4^- - \tilde k_5^-(2,5,3) - k_3^-} + \frac{\theta(-z)\tilde r_{5/3}\tilde r_{5/2} f_{3,54} f_{51,2}}{k_2^- - \tilde k_5^-(1,5,4) - k_1^-}. \tag{72}$$

The compact notation includes

$$f_{i,j} \equiv f_{\lambda_0}(\mathcal{M}_i^2, \mathcal{M}_j^2), \tag{73}$$

$$\tilde r_{5/i} \equiv \tilde r_\delta(x_{5/i}), \tag{74}$$

$$\mathcal{F}_{i,k,j} \equiv \mathcal{F}_{2\lambda_0}(i,k,j), \tag{75}$$

$$\tilde k_5^-(i,j,k) = [\kappa_5^{\perp 2} + \mu^2(i,j,k)]/k_5^+, \tag{76}$$

$$\kappa_5^\perp = \varepsilon(z)(\kappa_{13}^\perp - \kappa_{24}^\perp). \tag{77}$$

The mass terms with the $g_{\mu\nu}$ terms suppressed are

$$\tilde m_i^2 = \frac{4}{3} g^2 \int [x\kappa] \tilde r_\delta^2(x) f_{i,i'5'}^2 \frac{|j^+|^2}{k_i^{+2}} \frac{1}{x^2} \frac{x(\mathcal{M}^2 - m^2)}{m^2 - \mathcal{M}_i^2}, \tag{78}$$

for $i = 1,3$. $\mathcal{M}_1$ is given by Eq. (63) and $\mathcal{M}_3$ by Eq. (65). In both cases, $\mathcal{M}^2$ is given by Eq. (32), and the factor $(j^+/k_i^+)^2 = 4(1-x)$.

IV. SMALL-x BEHAVIOR

All small-x singularities of the eigenvalue Eq. (59) are contained in Eqs. (70), (71), and (78). We first discuss the exchange terms, then the mass terms, and finally the net effect of the interplay between these terms.

The analysis hinges on the properties of the energy of motion of a gluon with respect to the parent quark, $p_i^+ k_5^- = x_i \kappa_5^{\perp 2}/x_5$. p_i is the momentum of the parent quark i. The momentum $\kappa_5^\perp$ ranges under the integrals from 0 to ∞, while x_5 can reach 0 (in the mass terms, the integrals are over $\kappa^\perp$ and x). Appendixes B and C provide definitions of all variables used in the description of the integrands. The key difficulty is that the ratio of two variables of different kinds, $\kappa^\perp$ and x, varies quickly with a change of any one of them. This complexity is related to the power counting rules for the Hamiltonian densities on the LF [2]. But the analysis described here concerns only the relative motion of the effective particles and it is simplified by taking advantage of the NR limit after the finiteness of the small-x dynamics with the gluon mass ansatz is established.

The singularity in the effective gluon exchange term is tempered by the product of two vertex form factors *ff*. The form factors vanish exponentially fast when $\kappa_5^{\perp 2}/x_5 \to \infty$. This prevents x_5 from becoming small unless $\kappa_5^\perp$ vanishes at least as fast as $\sqrt{x_5}$. Therefore the measure of integration over transverse momenta is on the order of x_5 when $x_5 \to 0$ and it reduces the divergence to a logarithmic one. The logarithmic divergence is taken care of using the gluon mass ansatz. The mechanism of reducing singularities to only logarithmic ones does not work in the instantaneous interaction term $Z_\lambda(13,24)$ and in the terms in $V_\lambda(13,24)$ that come without *ff* in $\mathcal{F}_{2\lambda}$. But all the terms without *ff* are independent of μ^2 and the dx_5/x_5^2 and dx_5/x_5 singularities cancel out in them perturbatively [16].

In the fermion self-interactions an analogous pattern of the singularities occurs. But one has to also consider the size of m_0^2. The latter is determined by the size of the free ultraviolet-finite part of the quark mass counterterm in the initial Hamiltonian of Eq. (2). That size is related to an ansatz for a gluon mass term in the sectors $|Qg\rangle$ and $|\bar{Q}g\rangle$ in the eigenvalue equations for states with quantum numbers of a single fermion; see Appendix C. Eventually, the gluon mass ansatz leads to the result that the single-quark eigenvalue diverges logarithmically in the limit $\delta \to 0$, while the quark self-interaction in the quarkonium dynamics becomes finite. The self-interaction and effective gluon exchange, both finite due to the chosen behavior of the gluon mass ansatz, lead together to the harmonic potential which is described in the next section.

According to Appendix B, the dominant exchange terms in Eq. (70) read

$$\tilde{v}_0(13,24) = \theta(z)\tilde{v}_{+low} + \theta(-z)\tilde{v}_{-low}, \tag{79}$$

where $\tilde{v}_{+low}$ is given in Eq. (B19), and $\tilde{v}_{-low}$ in Eq. (B20). In the limit $x_5 \to 0$,

$$\tilde{v}_{+low} = f_{1,52} f_{53,4} \frac{\mu^2(2,5,3)}{q^{\perp 2} + \mu^2(2,5,3)}, \tag{80}$$

and

$$\tilde{v}_{-low} = f_{3,54} f_{51,2} \frac{\mu^2(1,5,4)}{q^{\perp 2} + \mu^2(1,5,4)}. \tag{81}$$

Since $q^{\perp 2}$ is on the order of $|z|$, one obtains the result that if μ^2 vanishes faster than $q^{\perp 2}$, i.e., faster than x_5, the potential produces a finite effect in the limit of $\delta \to 0$. In the denominators of Eqs. (80) and (81) there also appears $q_z^2 = (2mz)^2$, which is negligible in comparison to the leading terms on the order of z but can be included here on the basis of hindsight to take advantage of the NR nature of the quarks' motion with respect to each other. The larger is the quark mass m for fixed λ_0 and the smaller is Λ_{QCD}, the more accurate the NR picture actually becomes after the small-x divergences are removed. Writing $q_z = qt$, with $q = |\vec{q}|$, $t = \cos\theta$, the singular factor $1/x_5^2$ equals $4m^2/q_z^2 = (4m^2/q^2)t^{-2}$. The integration measure d^3q is proportional to q^2 and the small-x singularity is actually produced by the angle integration dt/t^2. μ^2 should vanish for $t \to 0$ in order to remove the singularity. An example of such behavior is used below to provide a constructive context for the steps that follow. The final result is not sensitive to the details of the example. Given that μ^2 vanishes faster than q^2, one can write

$$\mu^2(i,5,j) = c^2(i,5,j) q^2, \tag{82}$$

and determine behavior of $c(i,5,j)$ from the condition that

$$\tilde{c}(i,5,j) = \frac{c^2(i,5,j)}{1 + c^2(i,5,j)} \tag{83}$$

should vanish for $x_5 \to 0$.

The only information about the three-particle sector that is available in the relativistic construction of $\vec{q}$ and $c(i,5,j)$ are the $\perp$ and $+$ components of the momenta k_i, k_5, and k_j. Two physical constraints are used in defining a helpful vector $\vec{q}$: the definition must respect all kinematical LF symmetries (to render a boost-invariant description of quarkonia), and it must reduce to $q_z = 2mz$ for $z \to 0$ when $|\vec{k}_{13}|/m$ and $|\vec{k}_{24}|/m$ approach 0 in the NR limit. A geometrically motivated candidate for $\vec{q}$ is provided by the difference between the square of the free invariant mass of three effective particles in the state $|Q\bar{Q}g\rangle$ and the square of the invariant mass of the $Q\bar{Q}$ pair in this state. The difference reads

$$\mathcal{M}_{i5j}^2 - \mathcal{M}_{ij}^2 = \frac{\kappa_5^{\perp 2} + x_5^2 \mathcal{M}_{ij}^2}{x_5(1-x_5)}. \tag{84}$$

Multiplication by $x_5(1-x_5)$ produces an expression that tends in the limit of $x_5 \to 0$ to the three-momentum transfer squared that appears in the energy denominators in the small-x dynamics. The components of $\vec{q}$ are therefore defined as $q^\perp = \kappa_5^\perp$ and

$$q_z = z\mathcal{M}_{ij}. \tag{85}$$

Further analysis of all exchange terms shows that if the ansatz mass μ^2 behaves like

$$\mu^2 \sim x_5^{1+\delta_\mu} \tag{86}$$

(or like $q^2 x_5^{\delta_\mu}$) with $0<\delta_\mu<1/2$, the factors $\tilde{v}_{\pm low}$ of Eqs. (B19) and (B20) vanish in the limit $x_5 \to 0$ as $x_5^{\delta_\mu}$ independently of the terms in the energy denominators on the order of $x_5^{3/2}$ or smaller. Thus the gluon exchange term becomes finite when

$$c(i,5,j) = c(t) \tag{87}$$

and $c(t)$ is a function that behaves as

$$c(t) = c|t|^{\delta_\mu/2}, \tag{88}$$

for $t \to 0$, with c a constant.

With this ansatz, the quark mass terms also become finite in the limit $\delta \to 0$. Appendix C shows details of how m_0^2 is chosen in agreement with the physical picture explained in the Introduction and at the beginning of this section. Equation (78) gives $m_0^2 + \tilde{m}_i^2 = m^2 + \delta m_i^2$ with $i=1,3$ and

$$\delta m_i^2 = \frac{4\alpha_0}{3\pi^2} \int dx\, d^2\kappa^\perp\, \tilde{r}_\delta^2(x) f_{\lambda_0}^2(m^2, \mathcal{M}^2) \times \frac{\mathcal{M}^2 - m^2}{x^2} \left(\frac{1}{\mathcal{M}_0^2 - m^2} - \frac{1}{\mathcal{M}_i^2 - m^2} \right). \tag{89}$$

The function $\mathcal{M}_0^2$ is given by Eq. (C9) in terms of the gluon mass function μ_0^2 that must satisfy the condition (C10). The quark self-energies are positive if

$$\mu^2(i,5,j) > \mu_0^2. \tag{90}$$

A simple way to satisfy this condition is to set $\mu_0^2 = 0$. Then,

$$\delta m_i^2 = \frac{4\alpha}{3\pi^2} \int dx\, d^2\kappa^\perp\, f_{\lambda_0}^2(m^2, \mathcal{M}^2) \frac{1}{x^2} \times \frac{\mu^2(i',5',j)}{\mu^2(i',5',j) + (\kappa^{\perp 2} + x^2 m^2)/(1-x)}, \tag{91}$$

where $i=1$ and $j=3$ or vice versa. The factors r_δ are no longer needed.

The small-x regularization disappears from the quarkonium dynamics entirely. Finite phenomenological parameters that describe the small-x behavior of the gluon mass ansatz, such as δ_μ in Eq. (86), become responsible for the regularization of the exchange and self-interaction terms, preserving their distinct properties. The mass terms grow when δ_μ decreases, while the effective gluon exchange potential provides a negative contribution that increases in magnitude at similar rate and compensates the size of the masses at small momentum transfers. The net result is described in the next section.

V. $Q\bar{Q}$ SCHRÖDINGER EQUATION

The condition (51) validates the NR and weak-coupling limits after the small-x divergences are removed by the gluon mass ansatz. This section then identifies the leading structure in $H_{Q\bar{Q}}$ in formal order of α_0. The additional simplification in the case of small α_0 is that the dominant interaction in Eq. (59) becomes equal to the well-known Coulomb term and one can find the leading correction analytically.

Equation (59) can be re-written using the relative three-momentum variables described in Appendix B; see Eq. (B2). The integration measure is

$$\frac{dx_{24}\, d^2k_{24}^\perp}{x_2 x_4} = \frac{4 d^3\vec{k}_{24}}{\mathcal{M}_{24}}, \tag{92}$$

and Eq. (59) takes the form

$$\left[4(m^2 + \vec{k}_{13}^2) + \frac{\delta m_1^2}{x_1} + \frac{\delta m_3^2}{x_3} - (2m+B)^2 \right] \psi(\vec{k}_{13}) + \int \frac{d^3k_{24}}{(2\pi)^3 \sqrt{m^2 + k_{24}^2}} U(\vec{k}_{13}, \vec{k}_{24}) \psi(\vec{k}_{24}) = 0. \tag{93}$$

The mass corrections include now the $g_{\mu\nu}$ terms that were suppressed in the previous section, $\delta m_i^2 = \delta \tilde{m}_i^2 + \delta m_g^2$, and the potential is

$$U(\vec{k}_{13}, \vec{k}_{24}) = -\frac{4}{3} f_{13,24} 4\pi\alpha \left\{ 4\sqrt{\frac{x_1 x_2 x_3 x_4}{x_5^2}} \times [\theta(z)\tilde{v}_{+low} + \theta(-z)\tilde{v}_{-low}] + v_g \right\}, \tag{94}$$

where v_g denotes the $g_{\mu\nu}$ contribution in the exchange term.

Since the form factor $f_{13,24}$ cuts off changes of the relative momenta above λ_0 exponentially fast, one can focus on the eigenstates with lowest M^2 and take advantage of the conditions $|\vec{k}_{13}| \ll m$ and $|\vec{k}_{24}| \ll m$ that are satisfied in the entire domain of physically relevant probability distribution. For such states, one can expand Eq. (93) in powers of $\vec{k}/m$, with the exception of the form factors that are needed for convergence. The Coulomb force defines the momentum scale of the inverse of the quarkonium Bohr radius, $k_B = r_B^{-1} = \alpha_0 m/2$. When $\lambda_0 \gg k_B$, the form factor $f_{13,24}$ does not differ from 1 in the dynamically dominant region; $f_{13,24}$ matters only when one extends the expansion to high powers of $\vec{k}/m$. These would lead to divergent integrals with Coulomb wave functions and require counterterms [50,51]. The latter are not needed here and the lowest terms dominate [35]. The binding energy B is small in comparison to m. Writing the quarkonium mass as $M = 2m + B$ and neglecting $\sim B^2/m$, one obtains

$$\left[\frac{\vec{k}_{13}^2}{m}-B+\frac{\delta m_1^2}{2m}+\frac{\delta m_3^2}{2m}\right]\psi(\vec{k}_{13})$$
$$+\int\frac{d^3k_{24}}{(2\pi)^3}V_{Q\bar{Q}}(\vec{k}_{13}-\vec{k}_{24})\psi(\vec{k}_{24})=0. \tag{95}$$

The structure of $V_{Q\bar{Q}}$ and the size of the mass corrections δm_1^2 and δm_3^2 need explanation.

The two vertex form factors that appear inside the exchange and mass terms in Eq. (59), have arguments given in Appendix B in Eqs. (B8)–(B13). When one writes the product of the two vertex form factors in the NR limit as $\exp(-u^2)$, u reads

$$u=\sqrt{2}\,\frac{m}{\lambda_0}\,\frac{1}{t}\,\frac{q}{\lambda_0}. \tag{96}$$

The limit $m/\lambda_0\gg 1$ enforces $q\ll\lambda_0$, the more so the smaller is t. The Coulomb binding mechanism is intact for λ_0 as small as several times k_B [35], which is much smaller than m in the weak-coupling limit. Thus the momentum transfer q is much smaller than k_B in all terms that contain *ff*. These terms become then negligible in comparison to the Coulomb term, unless they have singularly small denominator factors for small t. That is the case for the mass and exchange terms when δ_μ becomes small. In the presence of the $g_{\mu\nu}$ contributions that were omitted in Sec. IV, these terms are found as follows.

The Hamiltonian $H_{Q\bar{Q}}$ has the structure

$$H_{Q\bar{Q}}=m^2+\delta m^2(ff,g+n,0)-\delta m^2(ff,g+n,\mu)$$
$$+f(1-ff)[(g+n,0)+z]+f(ff)[(g+n,\mu)+z], \tag{97}$$

where g denotes the $g_{\mu\nu}$ terms, n denotes the singular $n_\mu n_\nu$ terms, and z denotes the instantaneous terms. The gluon mass ansatz in energy denominators is indicated by an extra variable in the brackets, and 0 says that the gluon mass is 0. Note the difference between the last two terms in Eq. (97). The line with $f(1-ff)$ comes from the perturbative RGEP and has no gluon mass in it, while the line with $f(ff)$ comes from the exchange of effective gluons, its part $(1-f)(ff)$ being negligible, and it does involve the gluon mass ansatz in denominators. This difference is important because the next steps will show that the exchange terms with the gluon mass ansatz are relevant only to the spin-independent harmonic oscillator force that comes from the small-x region, while the Coulomb interaction will remain unchanged, i.e., not modified to a Yukawa interaction that one normally associates with a regular exchange of massive bosons. This happens because the mass ansatz enters only in the terms with factors *ff*, and these are able to produce a significant contribution only if they contain also factors that are singular in the small-x region, while the Coulomb term is regular there and comes mainly from 1 in the perturbative term $1-ff$, remaining unmodified. The last two terms can be re-arranged as

$$f(1-ff)[(g,0)+(n,0)+z]$$
$$+f(ff)[(g,0)+(g,\mu)-(g,0)+(n,\mu)+z]. \tag{98}$$

The contribution of $(n,0)+z$ in the first term vanishes in the leading NR limit; see Appendix B. Two of the terms with $(g,0)$ combine to $f(g,0)$, and reduce to the Coulomb term with the Breit-Fermi spin corrections. The remaining terms, with f in front also being equivalent to 1,

$$f(ff)[(g,\mu)-(g,0)+(n,\mu)+z], \tag{99}$$

add to the Coulomb term and produce together $V_{Q\bar{Q}}$ in Eq. (101). The mass terms can be re-written, in the same fashion, as

$$(ff)\,\delta m^2[(g,0)-(g,\mu)+(n,0)-(n,\mu)]. \tag{100}$$

Expressions (99) and (100) show that the exchange potential and the mass terms have similar structures with opposite signs. A change of variables from x and $\kappa^\perp$ to $q_z=xm$ and $q^\perp=\kappa^\perp$ in the mass terms produces integrals in which the factor *ff* ensures that $q=|\vec{q}|\ll m$ and one can again use the expansion in powers of the ratio of q/m. Since the integrands are symmetric functions of q_z, one can extend the integration to negative q_z and divide the result by 2, which produces the same integrands as in the exchange terms. Hence

$$V_{Q\bar{Q}}(\vec{q})=(1+BF)V_C(\vec{q})+W(\vec{q}), \tag{101}$$

where BF denotes the Breit-Fermi spin-dependent factors,

$$V_C(\vec{q})=-\frac{4}{3}\,\frac{4\pi\alpha}{q^2}, \tag{102}$$

$$W(\vec{q})=\frac{4}{3}4\pi\alpha\left[\frac{1}{\vec{q}^2}-\frac{1}{q_z^2}\right]\frac{\mu^2}{\mu^2+\vec{q}^2}\exp\left[-2\left(\frac{mq^2}{q_z\lambda_0^2}\right)^2\right], \tag{103}$$

with $\mu^2=\theta(z)\mu^2(2,5,3)+\theta(-z)\mu^2(1,5,4)$, and

$$\frac{\delta m_i^2}{m}=-\int\frac{d^3q}{(2\pi)^3}W(\vec{q}), \tag{104}$$

with μ^2 equal $\mu^2(1',5',3)$ for $i=1$ and $\mu^2(1,5',3')$ for $i=3$.

If the gluon mass ansatz is 0, $W=0$ and the quarkonium dynamics reduces to the same as in QED with additional color charge factor 4/3. A finite gluon mass ansatz introduces new dynamics which is discussed in the remaining part of this section.

W is large and negative when δ_μ is small. The exchange term tends to compensate the positive contribution of the mass terms. This can be made transparent by re-writing Eq. (95) as

$$\left[\frac{\vec{k}^2}{m}-B\right]\psi(\vec{k})+\int\frac{d^3q}{(2\pi)^3}(1+BF)V_C(\vec{q})\,\psi(\vec{k}-\vec{q})$$
$$+\int\frac{d^3q}{(2\pi)^3}W(\vec{q})[\psi(\vec{k}-\vec{q})-\psi(\vec{k})]=0. \quad (105)$$

There is no need to trace the small relativistic corrections before the main NR picture is identified. Only this picture is discussed below.

Since $|\vec{q}|$ in W is constrained to values much smaller than k_B, one can expand the wave function in the Coulomb region under the integral in the Taylor series and consider the lowest terms as candidates for the first approximation,

$$\psi(\vec{k}-\vec{q})=\psi(\vec{k})-q_i\frac{\partial}{\partial k_i}\psi(\vec{k})+\frac{1}{2}q_iq_j\frac{\partial^2}{\partial k_i\,\partial k_j}\psi(\vec{k})+\cdots. \quad (106)$$

The terms with odd powers of $\vec{q}$ average to 0. The bilinear terms contain q^2 times $(1-t^2)$ times $\cos^2\phi$, or $\sin^2\phi$, for $i=j=1,2$, respectively, and t^2, for $i=3$. The integral over ϕ produces π times a vector

$$\vec{w}(t)=(1-t^2,1-t^2,2t^2). \quad (107)$$

The variable q can be changed to u of Eq. (96), and introducing the constant

$$b=\frac{\sqrt{2m}}{\lambda_0^2}, \quad (108)$$

one obtains the vector

$$\vec{\tau}=\int_0^1 dt\,t(1-t^2)\vec{w}(t)\,\tau(t), \quad (109)$$

$$\tau(t)=\int_0^\infty du\,\frac{(b\mu/t)^2u^2}{(b\mu/t)^2+u^2}e^{-u^2}. \quad (110)$$

that appears in the resulting interaction term:

$$W_{Q\bar{Q}}=-\frac{4}{3}\frac{\alpha}{2\pi}b^{-3}\sum_{i=1}^{3}\tau_i\frac{\partial^2}{\partial k_i^2}. \quad (111)$$

The next nonvanishing terms in the Taylor expansion contain the fourth and higher even powers of $\vec{q}$. They are expected to be small in the momentum region dominated by the Coulomb dynamics and do not count around the bottom of the harmonic potential. The remaining question is if the harmonic approximation can be rotationally symmetric.

The interaction $W_{Q\bar{Q}}$ given by Eq. (111) is rotationally symmetric when all components of $\vec{\tau}$ are equal, or

$$\int_0^1 dt\,t(1-t^2)(1-3t^2)\,\tau(t)=0. \quad (112)$$

The function $\tau(t)$ depends on μ in a limited way because the integral over u in Eq. (110) extends only from 0 to about 1, b/t is large, and $b\mu/t\gg 1$ produces $\tau(t)=\beta$,

$$\beta=\sqrt{\pi}/4. \quad (113)$$

The behavior of $\tau(t)$ near $t=0$ does not matter because of the factor t in Eq. (112), and the condition (86) is of little consequence if μ^2 raises quickly from 0 at $t=0$. For any ansatz of Eq. (88) with a small δ_μ,

$$\tau(t)=\frac{c^2(t)}{1+c^2(t)}\beta, \quad (114)$$

which is equivalent to the constant β if $c(t)\gg 1$ for $t\neq 0$. The factorization feature is independent of the shape of the RGEP form factor f and provides an opportunity to fit μ analytically to satisfy Eq. (112). Suppose that Eq. (88) is valid and

$$\frac{c^2(t)}{1+c^2(t)}=c^2t^{\delta_\mu}(1-\rho t^2). \quad (115)$$

Equation (112) is satisfied when

$$\rho=\delta_\mu(8+\delta_\mu)/(2+\delta_\mu), \quad (116)$$

and

$$c^2(t)=\frac{c^2t^{\delta_\mu}(1-\rho t^2)}{1-c^2t^{\delta_\mu}(1-\rho t^2)} \quad (117)$$

leads to a rotationally symmetric harmonic oscillator potential. All components of $\vec{\tau}$ are equal $\tilde{\tau}$,

$$\tilde{\tau}=\beta c^2\left(\frac{1}{6}-\frac{\rho}{12}\right). \quad (118)$$

ρ varies between 0 and 17/25 when δ_μ varies between 0 and 1/2 in accord with Eq. (86). In the limit $\delta_\mu\to 0$, $\rho\to 0$ and $c^2(t)\sim c^2/(1-c^2)$ outside the area of very small t. In this example, the size of the coefficient functions $c(i,5,j)$ depends on the size of the constant c and grows to large values when $c\to 1$.

Every gluon mass ansatz μ^2 that is large outside the area of small t and vanishes abruptly for $t\to 0$ leads to a spherically symmetric harmonic oscillator potential with the spring constant

$$k=\frac{4}{3}\frac{\alpha}{\pi}b^{-3}\tilde{\tau}, \quad (119)$$

and

$$\tilde{\tau}\sim\beta/6. \quad (120)$$

The number β depends on the shape of the RGEP form factor f, which reflects the dependence of the effective dynamics on the RG scheme, but β is stable for f_{ab}s of similar shapes as functions of ab/λ^2; see Eq. (24).

The first approximation for heavy quarkonium dynamics in position space can be defined by the Fourier transform of the eigenvalue equation for $H_{Q\bar{Q}}$, with

$$\langle \vec{r}|k\rangle = \exp(i\vec{k}\vec{r}). \tag{121}$$

This transform exists only in the relative motion variables, since the motion of the quarkonium as a whole is described in a relativistic fashion and the relation between the relative motion of quarks and the motion of the bound state as a whole does not coincide for large speeds with the one known in NR theory. The Schrödinger equation reads

$$\left[2m - \frac{\Delta_r}{m} - \frac{4\alpha}{3}\left(\frac{1}{r} + BF\right) + \frac{k}{2}r^2\right]\psi(\vec{r}) = M\psi(\vec{r}), \tag{122}$$

where $k = m\omega^2/2$, and

$$\omega = \sqrt{\frac{4}{3}\frac{\alpha}{\pi}}\lambda\left(\frac{\lambda}{m}\right)^2\sqrt{\frac{\beta}{6\sqrt{2}}}. \tag{123}$$

The number $[\beta/(6\sqrt{2})]^{1/2}$ gives $(\pi/1152)^{1/4} \sim 0.23$, which is large enough for the frequency ω to fit into the ball park of phenomenologically plausible scales when one allows sufficiently large λ and α with some choice for m. The observed spectrum of charmonium is known to have an intermediate character between the Coulomb and harmonic oscillator spectra. But the key problem is to determine the size and direction of corrections that need to be included in order to compare the theory with data. The inclusion of light quarks requires quantitative understanding of the mechanism of chiral symmetry breaking in the effective particle approach and a comprehensive phenomenological analysis demands further advance in the theory.

With this reservation taken for granted, the simplicity and physically appealing content of Eqs. (122) and (123), especially the otherwise hard to explain quantum-mechanical effect of binding above the threshold of $2m$, deserve a clear summary of their apparent dependence on λ and no dependence on the regularizations introduced in the initial Hamiltonian. The effective dynamics of H_λ, and thus also $H_{Q\bar{Q}}$ for $\lambda = \lambda_0$, is not sensitive to the factors r_Δ because the RGEP equations produce H_λ near the end of the RG evolution they describe, and this is independent of the starting point at $\lambda = \infty$ where the regularization and corresponding counterterms are inserted in the QCD bare Hamiltonian. The fact that $H_{Q\bar{Q}}$ emerges as independent of the small-x regularization factors r_δ in the initial Hamiltonian is of different origin and follows from the color singlet nature of the quarkonium state. In the colorless state, effective quarks and gluons interact in a way that is no longer sensitive to the cutoff on gluon x at $x \sim \delta$ because the quark and antiquark are too close to each other to produce a long-range interaction of sufficient strength to introduce sensitivity to the small-x cutoff. In this situation, however, one may wonder what parameter is actually replacing δ in controlling the x-dependent factors, and why no such parameter is visible in Eqs. (122) and (123). The answer is that the effective gluon emission and re-absorption, and exchange, are limited in the region of $x \sim 0$ by how the gluons and quarks interact. This in turn is contained in the parameters of the mass ansatz for the effective gluons that move near the pair of quarks. These parameters limit x long before δ can matter. But the ansatz parameters turn out to be irrelevant for the final equation too, because what counts for x near 0 is a competition between how quarks self-interact and how they exchange effective gluons. The result of this competition depends on the gluon dispersion relation. But as long as the latter has the structure μ^2/x with nonzero μ^2, the details of μ^2 are secondary to the fact that μ^2/x is large and that f_λ modulates the interactions as for massless gluons and quite independently of the gluon mass ansatz. The result of this modulation provides Eqs. (122) and (123).

Thus, although there is no f_λ written explicitly in Eq. (122), the result is a consequence of the correlations implied by f_λ in the dynamics described by H_λ. These correlations become transparent after the energy of effective gluons is parametrized using the concept of a mass. Note, however, that the ansatz is introduced according to Eq. (1), and thus serves only to finesse the result that should be independent of μ^2 almost entirely for the true value of the coupling constant α at λs for which the state with two effective quarks approximates the full solution well. In this region, one should also expect that α and m depend on λ in such a way that solutions for the spectrum do not depend on λ. Unfortunately, this range cannot be estimated theoretically at the current level of approximation, and fourth-order terms are required to shed more light on the issue. Nevertheless, one should keep in mind that the parameter β depends somewhat on the function chosen for f_λ and its value given in Eq. (113) is only a good estimate of the result for a whole class of similar functions.

VI. CONCLUSION

The Coulomb interaction between quarks in heavy quarkonia is corrected by the potential well that is excavated by the one effective gluon exchange in the overlapping self-interaction gluon clouds of the quarks. At the bottom, the well shape is a quadratic function of the distance between the quarks. The resulting harmonic oscillator force plays the role of a confining one in a limited range. At distances much larger than the Bohr radius the quadratic approximation stops working. Emission of additional gluons and pairs of light quarks will further change the rate of growth of the potential. The size of these effects should be computable in the present approach by evaluating effective Hamiltonians order by order in a weak-coupling expansion and solving eigenvalue problems for them numerically.

The effective particle approach is of interest because it describes the relative motion of quarks independently of the speed of the quarkonium as a whole. This result is obtained at the price of setting up QCD in its Hamiltonian version in LF dynamics, with a host of difficulties in the renormalization program that had to be overcome. Further advances in the RGEP and methods of solving the eigenvalue equations for Hamiltonians H_λ are expected to reflect the well-known features of interactions of relativistic particles. The first approximation for $H_{Q\bar{Q}}$ can be expected to work well in the refined calculations because it appears to be largely independent in its structure from the details of the RGEP vertex form factors and the gluon mass ansatz. The first approximation also appears to involve the least possible degree of complexity as a basis around which a meaningful successive approximation scheme can emerge. A few percent accuracy in evaluating effective Hamiltonians is known to be achievable using essentially the same method in the case of elementary matrix models with asymptotic freedom and bound states.

Since the approach developed here is boost invariant, it can connect physical images of hadrons in different frames as soon as the hadron dynamics is understood in one of them. Although the light quarks are expected to behave differently from the heavy ones, one should note that the Schrödinger equation with H_λ does not lead to the spread of probability towards large relative momenta and large numbers of effective particles. The spread is halted because the interaction terms in H_λ contain form factors. These form factors are the reason for hope that the effective particle expansion may converge.

Aside from QCD, the same scheme for setting up and solving quantum field theory should be tested in the case of QED. There, the effective mass ansatz for virtual photons is much more restricted and small-x effects are of less significance. On the other hand, QED is not asymptotically free and its effective nature requires better understanding. The RGEP approach may help in defining QED as an effective theory. But one needs to first verify if perturbation theory with H_λ can produce covariant S-matrix in QED in orders higher than second.

APPENDIX A: REGULARIZED LF HAMILTONIAN OF QCD

The canonical LF Hamiltonians of gauge theories, similar to the Hamiltonians in the infinite momentum frame [68–70], are well known [71,72], and extensive literature exists on the lightlike axial gauges [73,74]. The Hamiltonian given below is further specified by inclusion of the ultraviolet and small-x regularization factors that render a computable operator. This means that H does not require a separate regularization prescription for evaluating loop integrals. The same regularized Hamiltonian was used in Ref. [32] but the quark terms needed here were not explicitly given there. H is supplied with counterterms $H_{\Delta\delta}$. Their structure is known from considerations similar to Ref. [2]. Details can be calculated using RGEP. The initial Lagrangian is

$$\mathcal{L}=\bar{\psi}(i\not{D}-m)\psi-\tfrac{1}{4}F^{\mu\nu a}F^{a}_{\mu\nu}, \tag{A1}$$

with one flavor of quarks of mass m,

$$F^{\mu\nu}=-i[D^\mu,D^\nu]/g, \tag{A2}$$

and

$$D_\mu=\partial_\mu+igA_\mu, \tag{A3}$$

where $A=A^a t^a$, $[t^a,t^b]=if^{abc}t^c$, and $\mathrm{Tr}(t^a t^b)=\delta^{ab}/2$. The classical Nether generator of evolution in x^+ takes the form (the Gauss law constraint is formally solved in $A^+=0$ gauge and the counterterms are added as the last term from hindsight),

$$H=H_{\psi^2}+H_{A^2}+H_{A^3}+H_{A^4}+H_{\psi A\psi}+H_{\psi AA\psi}+H_{[\partial AA]^2}+H_{[\partial AA](\psi\psi)}+H_{(\psi\psi)^2}+H_{\Delta\delta}, \tag{A4}$$

where each term is an integral over the LF hyperplane,

$$H_i=\int dx^-\,d^2x^\perp\,\mathcal{H}_i, \tag{A5}$$

and

$$\mathcal{H}_{\psi^2}=\frac{1}{2}\bar{\psi}\gamma^+\frac{-\partial^{\perp 2}+m^2}{i\partial^+}\psi, \tag{A6}$$

$$\mathcal{H}_{A^2}=-\frac{1}{2}A^\perp(\partial^\perp)^2A^\perp, \tag{A7}$$

$$\mathcal{H}_{A^3}=gi\partial_\alpha A^a_\beta[A^\alpha,A^\beta]^a, \tag{A8}$$

$$\mathcal{H}_{A^4}=-\frac{1}{4}g^2[A_\alpha,A_\beta]^a[A^\alpha,A^\beta]^a, \tag{A9}$$

$$\mathcal{H}_{\psi A\psi}=g\bar{\psi}\not{A}\psi, \tag{A10}$$

$$\mathcal{H}_{\psi AA\psi}=\frac{1}{2}g^2\bar{\psi}\not{A}\frac{\gamma^+}{i\partial^+}\not{A}\psi, \tag{A11}$$

$$\mathcal{H}_{[\partial AA]^2}=\frac{1}{2}g^2[i\partial^+A^\perp,A^\perp]^a\frac{1}{(i\partial^+)^2}[i\partial^+A^\perp,A^\perp]^a, \tag{A12}$$

$$\mathcal{H}_{[\partial AA](\psi\psi)}=g^2\bar{\psi}\gamma^+t^a\psi\frac{1}{(i\partial^+)^2}[i\partial^+A^\perp,A^\perp]^a, \tag{A13}$$

$$\mathcal{H}_{(\psi\psi)^2}=\frac{1}{2}g^2\bar{\psi}\gamma^+t^a\psi\frac{1}{(i\partial^+)^2}\bar{\psi}\gamma^+t^a\psi. \tag{A14}$$

A quantum Hamiltonian is introduced by expanding the fields into Fourier components at $x^+=0$ and imposing commutation relations on the latter. They define creation and annihilation operators for bare particles.

$$\psi=\sum_{\sigma c}\int[k][\chi_c u_{k\sigma}b_{k\sigma c}e^{-ikx}+\chi_c v_{k\sigma}d^\dagger_{k\sigma c}e^{ikx}]. \tag{A15}$$

The integration measure is

$$[k] = \frac{\theta(k^+)k^+ d^2k^\perp}{16\pi^3 k^+}, \quad (A16)$$

$$\{b_{k\sigma c}, b^\dagger_{k'\sigma' c'}\} = \{d_{k\sigma c}, d^\dagger_{k'\sigma' c'}\}$$
$$= 16\pi^3 k^+ \delta_{\sigma'_\sigma} \delta_{cc'} \delta^3(k-k'). \quad (A17)$$

$\delta^3(k-k') = \delta(k^+ - k'^+)\delta(k^1 - k'^1)\delta(k^2 - k'^2)$. The creation and annihilation operators have the power-counting dimension $1/k^\perp$ (the same result holds for gluons; see below). The spinors are given by $u_{k\sigma} = B(k,m)u_\sigma$ and $v_{k\sigma} = B(k,m)v_\sigma$, where $v_\sigma = Cu^*_\sigma = i\gamma^2 u^*_\sigma$. u_σ and v_σ are the spinors for the fermions at rest in the arbitrarily chosen frame of reference where the quantization procedure is introduced. The matrix $B(k,m)$ represents the LF boost that turns a particle with mass m at rest to a moving one that has the momentum k, $k^2 = m^2$,

$$B(k,m) = \frac{1}{\sqrt{k^+ m}}[\Lambda_+ k^+ + \Lambda_-(m + k^\perp \alpha^\perp)], \quad (A18)$$

where $\Lambda_\pm = \gamma_0\gamma^\pm/2$. This matrix mixes k^+ with m and $k^\perp$. But the second term, of the type $\sqrt{k^\perp/k^+}$ when one counts m and $k^\perp$ as similar, results only from writing the interaction terms in a way that is short and convenient in calculations. The independent degrees of freedom, $\psi_+ = \Lambda_+\psi$, contain only the parts proportional to $\sqrt{k^+/m}$. Thus ψ_+ has the dimension of $k^{\perp 2}\sqrt{k^+}b$, which in the position space on the LF leads to $(x^\perp\sqrt{x^-})^{-1}$ [2] if $b \sim 1/k^\perp$. The spinors at rest are

$$u_\sigma = \sqrt{2m}\begin{bmatrix}\chi_\sigma \\ 0\end{bmatrix}, \quad (A19)$$

$$v_\sigma = \sqrt{2m}\begin{bmatrix}0 \\ \xi_{-\sigma}\end{bmatrix}, \quad (A20)$$

where $\xi_{-\sigma} = -i\sigma_2\chi_\sigma = \sigma\chi_{-\sigma}$. The gluon field at $x^+ = 0$ is

$$A^\mu = \sum_{\sigma c}\int [k][t^c \varepsilon^\mu_{k\sigma} a_{k\sigma c} e^{-ikx} + t^c \varepsilon^{\mu *}_{k\sigma} a^\dagger_{k\sigma c} e^{ikx}], \quad (A21)$$

and the commutation relations read

$$[a_{k\sigma c}, a^\dagger_{k'\sigma' c'}] = 16\pi^3 k^+ \delta_{\sigma'_\sigma}\delta_{cc'}\delta^3(k-k'). \quad (A22)$$

a and $a^\dagger$ have the dimension $1/k^\perp$. The polarization four-vectors are introduced using LF boosts as for fermions,

$$\varepsilon^\mu_{k\sigma} = (\varepsilon^+_{k\sigma} = 0, \varepsilon^-_{k\sigma} = 2k^\perp \varepsilon^\perp_\sigma/k^+, \varepsilon^\perp_\sigma), \quad (A23)$$

except that the boosts are applied to the polarization vectors $\varepsilon^\mu_\sigma = (0,0,\varepsilon^\perp_\sigma)$ that correspond to the selected state of a gluon moving along the z axis [12]. Terms that contain the ratio $k^\perp/k^+$, which mix the transverse and longitudinal momenta, are again only a shorthand notation for writing interactions. The independent transverse field $A^\perp$ contains only polarization vectors $\varepsilon^\perp_\sigma$ that have dimension 1. $A^\perp$ has dimension of $k^{\perp 2}a$, which matches the required $1/x^\perp$ on the LF [2] when $a \sim 1/k^\perp$, as promised.

The kinetic energy operator for quarks, H_{ψ^2}, is given in Eq. (3), and for gluons, H_{A^2}, is given in Eq. (4). The triple-gluon interaction reads

$$H_{A^3} = \sum_{123}\int [123]\tilde{\delta}(p^\dagger - p)\tilde{r}_{\Delta\delta}(3,1)$$
$$\times [gY_{123}a^\dagger_1 a^\dagger_2 a_3 + gY^*_{123}a^\dagger_3 a_2 a_1]. \quad (A24)$$

The symbols introduced in this operator occur in all other terms and require explanation for completeness; see also Ref. [32]. The conservation of momentum in the interaction vertices is enforced by the factor

$$\tilde{\delta}(p^\dagger - p) = 16\pi^3\delta^3\left[\sum_{a^\dagger} p_{a^\dagger} - \sum_a p_a\right]. \quad (A25)$$

The regularization factors are given by

$$\tilde{r}_{\Delta\delta}(p,d) = r_{\Delta\delta}(p,d)r_{\Delta\delta}(p,p-d), \quad (A26)$$

where

$$r_{\Delta\delta}(p,d) = r_\Delta(\kappa^{\perp 2}_{d/p})r_\delta(x_{d/p})\theta(x_{d/p}). \quad (A27)$$

The symbol p refers to the parent momentum (half of the sum of all momenta of all particles coupled in a vertex), and d to the daughter particle momentum, i.e., the momentum of the particle emitted or absorbed in the vertex. The arguments of the regularization factors are defined by

$$\kappa^\perp_{d/p} = k^\perp_d - x_{d/p}k^\perp_p, \quad (A28)$$

$$x_{d/p} = k^+_d/k^+_p \equiv x_d/x_p. \quad (A29)$$

The functions used here are [32]

$$r_\Delta(\kappa^{\perp 2}) = \exp[-\kappa^{\perp 2}/\Delta^2], \quad (A30)$$

$$r_\delta(x) = \theta(x-\epsilon)x^\delta, \quad (A31)$$

and ϵ/δ tends to 0 when $\delta \to 0$. The gluon spin vertex factor reads

$$Y_{123} = if^{c_1c_2c_3}\left[\varepsilon^*_1\varepsilon^*_2\cdot\varepsilon_3\kappa - \varepsilon^*_1\varepsilon_3\cdot\varepsilon^*_2\kappa\frac{1}{x_{2/3}}\right.$$
$$\left. - \varepsilon^*_2\varepsilon_3\cdot\varepsilon^*_1\kappa\frac{1}{x_{1/3}}\right]. \quad (A32)$$

The simplified notation means: $\varepsilon \equiv \varepsilon^\perp$, $\kappa \equiv \kappa^\perp_{1/3}$. The quartic gluon vertex is

$$H_{A^4}=\sum_{1234}\int[1234]\tilde{\delta}(p^\dagger-p)\frac{g^2}{4}[\Xi_{A^41234}a_1^\dagger a_2^\dagger a_3^\dagger a_4+X_{A^41234}a_1^\dagger a_2^\dagger a_3a_4+\Xi^*_{A^41234}a_4^\dagger a_3a_2a_1], \quad \text{(A33)}$$

where

$$\Xi_{A^41234}=\frac{2}{3}[\tilde{r}_{1+2,1}\tilde{r}_{4,3}(\varepsilon_1^*\varepsilon_3^*\cdot\varepsilon_2^*\varepsilon_4-\varepsilon_1^*\varepsilon_4\cdot\varepsilon_2^*\varepsilon_3^*)\times f^{ac_1c_2}f^{ac_3c_4}+\tilde{r}_{1+3,1}\tilde{r}_{4,2}(\varepsilon_1^*\varepsilon_2^*\cdot\varepsilon_3^*\varepsilon_4-\varepsilon_1^*\varepsilon_4\cdot\varepsilon_2^*\varepsilon_3^*)\times f^{ac_1c_3}f^{ac_2c_4}+\tilde{r}_{3+2,3}\tilde{r}_{4,1}(\varepsilon_1^*\varepsilon_3^*\cdot\varepsilon_2^*\varepsilon_4-\varepsilon_3^*\varepsilon_4\cdot\varepsilon_2^*\varepsilon_1^*)\times f^{ac_3c_2}f^{ac_1c_4}], \quad \text{(A34)}$$

$$X_{A^41234}=\tilde{r}_{1+2,1}\tilde{r}_{3+4,3}(\varepsilon_1^*\varepsilon_3\cdot\varepsilon_2^*\varepsilon_4-\varepsilon_1^*\varepsilon_4\cdot\varepsilon_2^*\varepsilon_3)\times f^{ac_1c_2}f^{ac_3c_4}+[\tilde{r}_{3,1}\tilde{r}_{2,4}+\tilde{r}_{1,3}\tilde{r}_{4,2}](\varepsilon_1^*\varepsilon_2^*\cdot\varepsilon_3\varepsilon_4-\varepsilon_1^*\varepsilon_4\cdot\varepsilon_2^*\varepsilon_3)\times f^{ac_1c_3}f^{ac_2c_4}+[\tilde{r}_{3,2}\tilde{r}_{1,4}+\tilde{r}_{2,3}\tilde{r}_{4,1}](\varepsilon_1^*\varepsilon_2^*\cdot\varepsilon_3^*\varepsilon_4-\varepsilon_1^*\varepsilon_3\cdot\varepsilon_2^*\varepsilon_4)\times f^{ac_1c_4}f^{ac_2c_3}. \quad \text{(A35)}$$

The abbreviated notation for the regularization factors is $\tilde{r}_{p,d}\equiv\tilde{r}_{\Delta\delta}(p,d)$. Denoting $t^a_{ij}\equiv\chi^\dagger_{ic}t^a\chi_{jc}$, the quark-gluon coupling is given by

$$H_{\psi A\psi}=\sum_{123}\int[123]\tilde{\delta}(p^\dagger-p)\tilde{r}_{3,1}g[\overline{u_2}\not{\varepsilon}_1^*u_3t^1_{23}b_2^\dagger a_1^\dagger b_3-\overline{v_3}\not{\varepsilon}_1^*v_2t^1_{32}d_2^\dagger a_1^\dagger d_3+\overline{u_1}\not{\varepsilon}_3v_2t^3_{12}b_1^\dagger d_2^\dagger a_3+\text{H.c.}], \quad \text{(A36)}$$

where the spin vertex factors are

$$\overline{u_2}\not{\varepsilon}_1^*u_3=\sqrt{x_3/x_2}\chi_2^\dagger\left[i(\kappa^\perp_{1/3}\varepsilon_1^{*\perp})^3\sigma^3+\frac{x_2+x_3}{x_1}\kappa^\perp_{1/3}\varepsilon_1^{*\perp}-m_3\frac{x_1}{x_3}\sigma^\perp\varepsilon_1^{*\perp}\sigma^3\right]\chi_3, \quad \text{(A37)}$$

$$\overline{v_3}\not{\varepsilon}_1^*v_2=\sqrt{x_3/x_2}\xi^\dagger_{-3}\left[-i(\kappa^\perp_{1/3}\varepsilon_1^{*\perp})^3\sigma^3+\frac{x_2+x_3}{x_1}\kappa^\perp_{1/3}\varepsilon_1^{*\perp}-m_3\frac{x_1}{x_3}\sigma^3\sigma^\perp\varepsilon_1^{*\perp}\right]\xi_{-2}, \quad \text{(A38)}$$

$$\overline{u_1}\not{\varepsilon}_3v_2=\sqrt{x_3/x_1}\sqrt{x_3/x_2}\chi_1^\dagger\left[-i(\kappa^\perp_{1/3}\varepsilon_3^\perp)^3+\frac{x_1-x_2}{x_3}\kappa^\perp_{1/3}\varepsilon_3^\perp\sigma^3-m_1\sigma^\perp\varepsilon_3^\perp\right]\xi_{-2}. \quad \text{(A39)}$$

The instantaneous fermion interaction reads

$$H_{\psi AA\psi}=\sum_{1234}\int[1234]\tilde{\delta}(p^\dagger-p)(g^2/2)2\sqrt{x_1x_4}\cdot\{\}, \quad \text{(A40)}$$

where the curly brackets {} contain the operators ordered according to the rule $b^\dagger d^\dagger a^\dagger adb$;

$$\begin{aligned}\{\}=&\tilde{r}_{1+2,1}\tilde{r}_{3+4,3}\left[t^{23}_{14}\frac{\chi_1^\dagger[2*3]\chi_4}{x_1+x_2}b_1^\dagger a_2^\dagger a_3b_4+t^{23}_{14}\frac{\xi^\dagger_{-1}[23*]\xi_{-4}}{x_1+x_2}d_4^\dagger a_3^\dagger a_2d_1\right]\\&+[\tilde{r}_{1,2}\tilde{r}_{4,3}+\tilde{r}_{2,1}\tilde{r}_{3,4}]\left[t^{23}_{14}\frac{\chi_1^\dagger[23*]\chi_4}{x_1-x_2}b_1^\dagger a_3^\dagger a_2b_4+t^{23}_{14}\frac{\xi^\dagger_{-1}[2*3]\xi_{-4}}{x_1-x_2}d_4^\dagger a_2^\dagger a_3d_1\right]\\&+\left[\tilde{r}_{3,4}\tilde{r}_{1+2,1}t^{23}_{14}\frac{\chi_1^\dagger[2*3]\sigma^3\xi_{-4}}{x_1+x_2}b_1^\dagger d_4^\dagger a_2^\dagger a_3+\text{H.c.}\right]+\left[\tilde{r}_{2,1}\tilde{r}_{3+4,3}t^{23}_{14}\frac{\chi_1^\dagger[23*]\sigma*3\xi_{-4}}{x_1-x_2}b_1^\dagger d_4^\dagger a_3^\dagger a_2+\text{H.c.}\right]\\&+\left[\frac{1}{2}\left([\tilde{r}_{1,2}\tilde{r}_{3,4}+\tilde{r}_{2,1}\tilde{r}_{4,3}]t^{23}_{14}\frac{\chi_1^\dagger[23]\sigma^3\xi_{-4}}{x_1-x_2}+[\tilde{r}_{1,3}\tilde{r}_{2,4}+\tilde{r}_{3,1}\tilde{r}_{4,2}]t^{32}_{14}\frac{\chi_1^\dagger[32]\sigma^3\xi_{-4}}{x_1-x_3}\right)b_1^\dagger d_4^\dagger a_2a_3+\text{H.c.}\right]\\&+\left[\frac{1}{2}\left(\tilde{r}_{4,3}\tilde{r}_{1+2,1}t^{23}_{14}\frac{\chi_1^\dagger[2*3*]\chi_4}{x_1+x_2}+\tilde{r}_{4,2}\tilde{r}_{1+3,1}t^{32}_{14}\frac{\chi_1^\dagger[3*2*]\chi_4}{x_1+x_3}\right)b_1^\dagger a_2^\dagger a_3^\dagger b_4+\text{H.c.}\right]\\&+\left[\frac{1}{2}\left(\tilde{r}_{1,2}\tilde{r}_{3+4,4}t^{23}_{14}\frac{\xi^\dagger_{-1}[2*3*]\xi_{-4}}{x_1-x_2}+\tilde{r}_{1,3}\tilde{r}_{2+4,4}t^{32}_{14}\frac{\xi^\dagger_{-1}[3*2*]\xi_{-4}}{x_1-x_3}\right)d_4^\dagger a_2^\dagger a_3^\dagger d_1+\text{H.c.}\right]. \quad \text{(A41)}\end{aligned}$$

The symbols mean: $t_{ij}^{ab}=\chi_{ic}^{\dagger}t^a t^b \chi_{jc}$, and $[\alpha\beta]=\varepsilon_\alpha^\perp \varepsilon_\beta^\perp + i(\varepsilon_\alpha^\perp \times \varepsilon_\beta^\perp)^3\sigma^3$. The star, *, means that the corresponding polarization vector is complex conjugated: $*\to\varepsilon^*$. The quartic gluon term with derivative reads

$$H_{[\partial AA]^2}=\sum_{1234}\int [1234]\tilde{\delta}(p^\dagger-p)g^2[(\Xi_{[\partial AA]^2 1234}a_1^\dagger a_2^\dagger a_3^\dagger a_4 + \text{H.c.})+X_{[\partial AA]^2\ 1234}a_1^\dagger a_2^\dagger a_3 a_4], \tag{A42}$$

where

$$\begin{aligned}\Xi_{[\partial AA]^2 1234}=&-\frac{1}{6}\bigg[\tilde{r}_{1+2,1}\tilde{r}_{4,3}\varepsilon_1^*\varepsilon_2^*\cdot\varepsilon_3^*\varepsilon_4 \\ &\times\frac{(x_1-x_2)(x_3+x_4)}{(x_1+x_2)^2}f^{ac_1c_2}f^{ac_3c_4} \\ &+\tilde{r}_{1+3,1}\tilde{r}_{4,2}\varepsilon_1^*\varepsilon_3^*\cdot\varepsilon_2^*\varepsilon_4 \\ &\times\frac{(x_1-x_3)(x_2+x_4)}{(x_1+x_3)^2}f^{ac_1c_3}f^{ac_2c_4} \\ &+\tilde{r}_{3+2,3}\tilde{r}_{4,1}\varepsilon_3^*\varepsilon_2^*\cdot\varepsilon_1^*\varepsilon_4 \\ &\times\frac{(x_3-x_2)(x_1+x_4)}{(x_3+x_2)^2}f^{ac_3c_2}f^{ac_1c_4}\bigg],\end{aligned} \tag{A43}$$

$$\begin{aligned}X_{[\partial AA]^2 1234}=&\frac{1}{4}\bigg[\tilde{r}_{1+2,1}\tilde{r}_{3+4,3}\varepsilon_1^*\varepsilon_2^*\cdot\varepsilon_3\varepsilon_4 \\ &\times\frac{(x_1-x_2)(x_3-x_4)}{(x_1+x_2)^2}f^{ac_1c_2}f^{ac_3c_4} \\ &-[\tilde{r}_{3,1}\tilde{r}_{2,4}+\tilde{r}_{1,3}\tilde{r}_{4,2}]\varepsilon_1^*\varepsilon_3\cdot\varepsilon_2^*\varepsilon_4 \\ &\times\frac{(x_1+x_3)(x_2+x_4)}{(x_2-x_4)^2}f^{ac_1c_3}f^{ac_2c_4} \\ &-[\tilde{r}_{3,2}\tilde{r}_{1,4}+\tilde{r}_{2,3}\tilde{r}_{4,1}]\varepsilon_1^*\varepsilon_4\cdot\varepsilon_2^*\varepsilon_4 \\ &\times\frac{(x_2+x_3)(x_1+x_4)}{(x_1-x_4)^2}f^{ac_1c_4}f^{ac_2c_3}\bigg].\end{aligned} \tag{A44}$$

The instantaneous gluon interaction between quarks and gluons reads

$$H_{[\partial AA](\psi\psi)}=\sum_{1234}\int [1234]\tilde{\delta}(p^\dagger-p)g^2 i f^{a12}t_{34}^a 2\sqrt{x_3x_4}\{\}, \tag{A45}$$

where the brackets {} contain

$$\begin{aligned}\{\}=&\varepsilon_1^*\varepsilon_2^*\frac{x_2-x_1}{2(x_1+x_2)^2}[\tilde{r}_{1+2,1}\tilde{r}_{3+4,3}\xi_{-3}^\dagger\sigma^3\chi_4 a_1^\dagger a_2^\dagger d_3 b_4 \\ &+\tilde{r}_{1+2,1}\tilde{r}_{4,3}\chi_3^\dagger\chi_4 b_3^\dagger a_1^\dagger a_2^\dagger b_4-\tilde{r}_{1+2,1}\tilde{r}_{3,4}\xi_{-3}^\dagger\xi_{-4}d_4^\dagger a_1^\dagger a_2^\dagger d_3] \\ &-\varepsilon_1^*\varepsilon_2\frac{x_1+x_2}{(x_3+x_4)^2}\tilde{r}_{3+4,3}\tilde{r}_{2,1}\chi_3^\dagger\sigma^3\xi_{-4}b_3^\dagger d_4^\dagger a_1^\dagger a_2 \\ &+\varepsilon_1^*\varepsilon_2\frac{x_1+x_2}{(x_1-x_2)^2}[(\tilde{r}_{2,1}\tilde{r}_{4,3}+\tilde{r}_{1,2}\tilde{r}_{3,4})\xi_{-3}^\dagger\xi_{-4}d_4^\dagger a_1^\dagger a_2 d_3 \\ &-(\tilde{r}_{2,1}\tilde{r}_{3,4}+\tilde{r}_{1,2}\tilde{r}_{4,3})\chi_3^\dagger\chi_4 b_3^\dagger a_1^\dagger a_2 b_4]+\text{H.c.}\end{aligned} \tag{A46}$$

Finally, the instantaneous gluon interaction between quarks reads

$$H_{(\psi\psi)^2}=\sum_{1234}\int [1234]\tilde{\delta}(p^\dagger-p)\frac{g^2}{2}4\sqrt{x_1x_2x_3x_4}\{\}, \tag{A47}$$

where the brackets {} contain

$$\begin{aligned}\{\}=&-\frac{1}{2}\bigg[\frac{\chi_1^\dagger\chi_2\chi_3^\dagger\chi_4}{(x_1-x_2)^2}t_{12}^a t_{34}^a[\tilde{r}_{1,2}\tilde{r}_{4,3}+\tilde{r}_{2,1}\tilde{r}_{3,4}]-\frac{\chi_3^\dagger\chi_2\chi_1^\dagger\chi_4}{(x_3-x_2)^2}t_{32}^a t_{14}^a[\tilde{r}_{3,2}\tilde{r}_{4,1}+\tilde{r}_{2,3}\tilde{r}_{1,4}]\bigg]b_1^\dagger b_3^\dagger b_2 b_4 \\ &+\frac{1}{2}\bigg[\frac{\xi_{-2}^\dagger\xi_{-1}\xi_{-4}^\dagger\xi_{-3}}{(x_1-x_2)^2}t_{21}^a t_{43}^a[\tilde{r}_{1,2}\tilde{r}_{4,3}+\tilde{r}_{2,1}\tilde{r}_{3,4}]-\frac{\xi_{-2}^\dagger\xi_{-3}\xi_{-4}^\dagger\xi_{-1}}{(x_3-x_2)^2}t_{23}^a t_{41}^a[\tilde{r}_{3,2}\tilde{r}_{4,1}+\tilde{r}_{2,3}\tilde{r}_{1,4}]\bigg]d_1^\dagger d_3^\dagger d_2 d_4 \\ &+\left(\bigg[\frac{\chi_1^\dagger\chi_2\chi_3^\dagger\sigma^3\xi_{-4}}{(x_1-x_2)^2}t_{12}^a t_{34}^a\tilde{r}_{2,1}\tilde{r}_{3+4,3}-\frac{\chi_3^\dagger\chi_2\chi_1^\dagger\sigma^3\xi_{-4}}{(x_3-x_2)^2}t_{32}^a t_{14}^a\tilde{r}_{2,3}\tilde{r}_{1+4,1}\bigg]b_1^\dagger b_3^\dagger d_4^\dagger b_2+\text{H.c.}\right) \\ &-\left(\bigg[\frac{\chi_1^\dagger\sigma^3\xi_{-2}\xi_{-4}^\dagger\xi_{-3}}{(x_1+x_2)^2}t_{12}^a t_{43}^a\tilde{r}_{4,3}\tilde{r}_{1+2,1}-\frac{\chi_1^\dagger\sigma^3\xi_{-3}\xi_{-4}^\dagger\xi_{-2}}{(x_1+x_3)^2}t_{13}^a t_{42}^a\tilde{r}_{4,2}\tilde{r}_{1+3,1}\bigg]b_1^\dagger d_2^\dagger d_3^\dagger d_4+\text{H.c.}\right) \\ &-2\frac{\chi_1^\dagger\chi_2\xi_{-4}^\dagger\xi_{-3}}{(x_1-x_2)^2}t_{12}^a t_{43}^a[\tilde{r}_{1,2}\tilde{r}_{4,3}+\tilde{r}_{2,1}\tilde{r}_{3,4}]b_1^\dagger d_3^\dagger d_4 b_2+2\frac{\chi_1^\dagger\sigma^3\xi_{-3}\xi_{-4}^\dagger\sigma^3\chi_2}{(x_1+x_3)^2}t_{13}^a t_{42}^a\tilde{r}_{1+3,1}\tilde{r}_{2+4,2}b_1^\dagger d_3^\dagger d_4 b_2.\end{aligned} \tag{A48}$$

Useful color identities are: $t^a t^b t^a = -t^a/(2N_c)$, $t^a t^b + t^b t^a = \delta^{ab}/N_c + d^{abc} t^c$, $d^{abc} d^{abd} = [(N_c^2-1)/N_c]\delta^{cd}$, $f^{abc} t^b t^c = i(N_c/2) t^a$.

APPENDIX B: $Q_\lambda \bar{Q}_\lambda$ INTERACTION

Several factors are needed to estimate the small-x behavior of the potential kernel $\tilde{v}_0(13,24)$ in Eq. (71). Momenta are labeled according to Figs. 1 and 2.

$$f_{13,24} = \exp[-(\mathcal{M}_{13}^2 - \mathcal{M}_{24}^2)^2/\lambda^4], \tag{B1}$$

where

$$\mathcal{M}_{ij}^2 = 4(m^2 + |\vec{k}_{ij}|^2) = \frac{\kappa_{ij}^{\perp 2} + m^2}{x_i x_i}, \tag{B2}$$

with

$$k_{ij}^\perp = \kappa_{ij}^\perp, \tag{B3}$$

$$k_{ij}^3 = (x_i - 1/2)\mathcal{M}_{ij}, \tag{B4}$$

$$\mathcal{M}_{13}^2 - \mathcal{M}_{24}^2 = 4(\vec{k}_{13} + \vec{k}_{24})\vec{q}, \tag{B5}$$

where

$$\vec{q} = \vec{k}_{13} - \vec{k}_{24}, \tag{B6}$$

is the momentum transfer that goes over to the standard one in the NR limit,

$$\mathcal{M}_{13}^2 - \mathcal{M}_{24}^2 = \frac{2\kappa_{13}^\perp q^\perp - q^{\perp 2} - z(1-2x_1+z)\mathcal{M}_{13}^2}{(x_1-z)(x_3+z)}. \tag{B7}$$

In the last term in Eq. (71), the form factors $f_{1,52} f_{53,4}$ have arguments

$$\mathcal{M}_{52}^2 - m^2 = \frac{x_1}{x_1 - z}\mathcal{D}_1, \tag{B8}$$

where

$$\mathcal{D}_1 \equiv \frac{x_1}{|z|}\left(q^\perp - \frac{z}{x_1}\kappa_{13}^\perp\right)^2 + m^2\frac{|z|}{x_1}, \tag{B9}$$

and,

$$\mathcal{M}_{53}^2 - m^2 = \mathcal{D}_3, \tag{B10}$$

where

$$\mathcal{D}_3 = \frac{x_3}{|z|}\left(q^\perp + \frac{z}{x_3}\kappa_{13}^\perp\right)^2 + m^2\frac{|z|}{x_3}, \tag{B11}$$

while the form factors $f_{3,54} f_{51,2}$ have the arguments

$$\mathcal{M}_{54}^2 - m^2 = \frac{x_3}{x_3 + z}\mathcal{D}_3, \tag{B12}$$

and

$$\mathcal{M}_{51}^2 - m^2 = \mathcal{D}_1. \tag{B13}$$

The last term in Eq. (71) can be written with the coefficient $1 = (1 - f_{13,24}) + f_{13,24}$ but only the second term counts at small z because $(1 - f_{13,24})$ is proportional to the momentum transfer squared. The factor $f_{13,24}$ becomes common to all terms in Eq. (71) and is taken out in front. The LF instantaneous term can be split into the part ff that joins the low-energy exchange and $1 - ff$ that goes with the high-energy exchange. This way one obtains

$$\left(\frac{\kappa_{13}^{\perp 2} + m_0^2}{x_1 x_3} + \frac{\tilde{m}_1^2}{x_1} + \frac{\tilde{m}_3^2}{x_3} - M^2\right)\tilde{\psi}(\kappa_{13}^\perp, x_1) - \frac{4\alpha}{3\pi^2}\int dx_2\, d^2\kappa_{24}^\perp \sqrt{\frac{x_1 x_3}{x_2 x_4}}\frac{f_{13,24}}{x_5^2}\tilde{v}_0(13,24)\tilde{\psi}(\kappa_{24}^\perp, x_2) = 0, \tag{B14}$$

where

$$\tilde{v}_0(13,24) = \theta(z)\tilde{r}_{5/1}\tilde{r}_{5/4}(\tilde{v}_{+high} + \tilde{v}_{+low}) + \theta(-z)\tilde{r}_{5/3}\tilde{r}_{5/2}(\tilde{v}_{-high} + \tilde{v}_{-low}). \tag{B15}$$

The gluon mass ansatz contributes to the low-energy exchange terms only. In terms of the invariant masses from Eqs. (B8)–(B13),

$$\tilde{v}_{+high} = \frac{f_{1,52} f_{53,4} - 1}{2} \times \left\{\frac{x_1^2 + x_4^2}{x_1 x_4}\frac{(\mathcal{M}_{52}^2 - m^2)(\mathcal{M}_{53}^2 - m^2)}{(\mathcal{M}_{52}^2 - m^2)^2 + (\mathcal{M}_{53}^2 - m^2)^2} - 1\right\}, \tag{B16}$$

$$\tilde{v}_{-high} = \frac{f_{3,54} f_{51,2} - 1}{2} \times \left\{\frac{x_3^2 + x_2^2}{x_3 x_2}\frac{(\mathcal{M}_{54}^2 - m^2)(\mathcal{M}_{51}^2 - m^2)}{(\mathcal{M}_{54}^2 - m^2)^2 + (\mathcal{M}_{51}^2 - m^2)^2} - 1\right\}. \tag{B17}$$

These have the same limit when $z \to 0$ for fixed $q^\perp$,

$$\lim_{z\to 0}\tilde{v}_{+high} = \lim_{z\to 0}\tilde{v}_{-high} = \frac{f_{3,54} f_{51,2} - 1}{2}\frac{x_1 - x_3}{x_1 x_3(x_1^2 + x_3^2)}\frac{(q^\perp - \kappa_{13}^\perp)^2 - \kappa_{13}^{\perp 2}}{q^{\perp 2}} z + o(z^2). \tag{B18}$$

The terms on the order of z^2 and higher are finite when divided by the square of $x_5=|z|$. Terms linear in z produce an integral convergent in the sense of principal value [2,16]. When $q^\perp\sim\sqrt{z}\to 0$, $d^2\kappa_{24}^\perp$ removes one power of z from the denominator in Eq. (70), while $\tilde{v}_{\pm high}$ vanish for $z\to 0$. The contributions of $\tilde{v}_{\pm high}$ are $\vec{k}^2/m^2$ times smaller than the dominant terms and can be ignored in the first approximation. One can see this by integrating $\tilde{v}_{\pm high}$ with a Coulomb wave function.

The low-energy terms read

$$\tilde{v}_{+low}=\frac{f_{1,52}f_{53,4}}{4}\left[2-\frac{(\mathcal{M}_{53}^2-m^2)/x_4-\mu^2(2,5,3)/x_5}{(\mathcal{M}_{52}^2-m^2)/x_1+\mu^2(2,5,3)/x_5}-\frac{(\mathcal{M}_{52}^2-m^2)/x_1-\mu^2(2,5,3)/x_5}{(\mathcal{M}_{53}^2-m^2)/x_4+\mu^2(2,5,3)/x_5}\right], \tag{B19}$$

$$\tilde{v}_{-low}=\frac{f_{3,54}f_{51,2}}{4}\left[2-\frac{(\mathcal{M}_{51}^2-m^2)/x_2-\mu^2(1,5,4)/x_5}{(\mathcal{M}_{54}^2-m^2)/x_3+\mu^2(1,5,4)/x_5}-\frac{(\mathcal{M}_{54}^2-m^2)/x_3-\mu^2(1,5,4)/x_5}{(\mathcal{M}_{51}^2-m^2)/x_2+\mu^2(1,5,4)/x_5}\right]. \tag{B20}$$

APPENDIX C: MASS TERMS

The mass terms with $i=1,3$ in the eigenvalue Eq. (B14) are given in Eq. (89), with $\mathcal{M}_1^2$ given in Eq. (63) and $\mathcal{M}_3^2$ in Eq. (65). m_0^2 originates from Eq. (28) with $\lambda=\lambda_0$. Namely, the quark mass counterterm in X of Eq. (2) adds $\delta m_{\Delta\delta}^2$ to the original mass parameter m^2 in Eq. (3) and the free ultraviolet-finite part of the counterterm is such that m_0^2 appears in Eq. (70). The condition on m_0^2 that the eigenstates of H_{λ_0} with quantum numbers of a single quark have eigenvalues growing to infinity is fulfilled below by representing gluon interactions in the case of the single quark state by a new gluon mass ansatz. The resulting value of m_0^2 enters into the quarkonium dynamics. The determination of the ultraviolet-finite part of the mass counterterm in X in Eq. (2) is thus based on the picture that comes out from simultaneous consideration of two eigenvalue equations, one for the state with quantum numbers of a single quark (or an antiquark, the result is the same), and another one for the quarkonium. The key physical assumption made in the comparison is that the binding of effective quarks in the quarkonium state occurs at the expense of change in their individual structure. While the buildup of self-interacting clouds of gluons around single quarks leads to the infinite quark masses, in the case of a colorless pair the main parts of the gluon clouds can recombine into a colorless object that may fly out of the region of strong interaction with the quarks, leaving behind only the minimal remnants of the gluon clouds required to form the quarkonium eigenstate with a finite mass. The new finite balance is described using the gluon mass ansatz parameter δ_μ. The finite balance can be achieved because the quark-antiquark state looks neutral from large distances and does not continue to generate gluons over infinite distances along the LF. This scenario is partly similar to the one originally developed in the LF dynamics in Refs. [42,43], and studied in Refs. [44,45]. The main differences are related to the fact that the physical picture that emerges here in the finite effective theory with the gluon mass ansatz relies on the phenomenological parameter δ_μ. A formal cutoff parameter δ^+ of the canonical theory, the coupling coherence phenomenon that may work over many scales of an ultraviolet cutoff, and the condition of transverse locality are not employed in the new picture. Instead, the present scenario can be studied in higher orders of perturbation theory according to the known rules [30,32] that explicitly preserve the boost invariance, cluster decomposition property, and unitary connection with the initial theory.

The eigenstate of H_{λ_0} with a single quark quantum numbers and momentum p with components p^+ and $p^\perp$ has the eigenvalue

$$p^-=(p^{\perp 2}+\tilde{m}^2)/p^+, \tag{C1}$$

and the decomposition in the effective particle basis,

$$|p\rangle=|Q_{\lambda_0}\rangle+|Q_{\lambda_0}g_{\lambda_0}\rangle+\cdots. \tag{C2}$$

The new gluon mass ansatz is introduced in the quark-gluon component. It is different than in the quarkonium case because the states have different quantum numbers and are made of different numbers of effective particles. Dropping the subscript λ_0 as in Eq. (54), the eigenvalue problem is written as

$$(T_q+\tilde{T}_g)|Qg\rangle+Y|Q\rangle=E|Qg\rangle,$$
$$Y|Qg\rangle+T_q|Q\rangle=E|Q\rangle. \tag{C3}$$

The new ansatz enters through the kinetic energy $\tilde{T}_g$, which contains

$$\tilde{\mu}_{\lambda_0}^2=\mu_{\lambda_0}^2+\mu_Q^2(x,\kappa^\perp), \tag{C4}$$

where x and $\kappa^\perp$ refer to the relative motion of the effective gluon with respect to the quark. The operator R from Eq. (57) is now replaced by the one with $\hat{\mathcal{P}}$ that projects on the single effective quark basis state with kinematical momentum components p^+ and $p^\perp$. In the perturbative expansion in g, only the second term on the right-hand side of Eq. (C4) contributes in orders up to g^2. Thanks to the boost invariance, the resulting eigenvalue condition reduces to an equation for $\tilde{m}^2$, which is independent of p,

$$\tilde{m}^2=m_0^2-\frac{4\alpha_0}{3\pi^2}\int dx\, d^2\kappa^\perp\,\tilde{r}_\delta^2(x)\times f_{\lambda_0}^2(m^2,\mathcal{M}^2)\frac{1}{x^2}\frac{\mathcal{M}^2-m^2}{\mathcal{M}_Q^2-m^2}, \tag{C5}$$

$$\mathcal{M}_Q^2=[\kappa^{\perp 2}+\mu_Q^2(x,\kappa^\perp)]/x+(\kappa^{\perp 2}+m^2)/(1-x). \tag{C6}$$

For $\widetilde{m}^2$ to be positive and growing to infinity when $\delta\to 0$, one can write m_0^2 in the integral form,

$$m_0^2=m^2+\frac{4\alpha_0}{3\pi^2}\int dx\, d^2\kappa^\perp\, \widetilde{r}^{\,2}_\delta(x) f^2_{\lambda_0}(m^2,\mathcal{M}^2)\frac{1}{x^2}\frac{\mathcal{M}^2-m^2}{\mathcal{M}_0^2-m^2}, \tag{C7}$$

with some function $\mathcal{M}_0^2$ that satisfies the condition

$$\mathcal{M}_Q^2>\mathcal{M}_0^2>m^2. \tag{C8}$$

This condition can be satisfied by writing

$$\mathcal{M}_0^2=[\kappa^{\perp 2}+\mu_0^2(x,\kappa^\perp)]/x+(\kappa^{\perp 2}+m^2)/(1-x), \tag{C9}$$

and assuming that

$$\mu_Q^2>\mu_0^2\geqslant 0. \tag{C10}$$

As long as the difference $\mu_Q^2-\mu_0^2$ does not vanish for $x\to 0$, the single quark mass will tend to ∞ when $\delta\to 0$. But this may easily happen here because the larger is the gluon mass ansatz μ_Q^2, the stronger the single quark mass eigenvalue diverges in the limit $\delta\to 0$, while μ_0^2 remains free to vanish in the limit $x\to 0$ and lead to a finite mass contribution in the quarkonium dynamics. Using Eq. (C7) for m_0^2 one obtains Eq. (89).

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